The functional equation for certain twisted Rankin convolutions and an application to the spinor L-function

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Contents

§ 1 Notation and well known constructions

Given a matrix M let M^{tr} denote the transposed matrix, whereas $\mathrm{tr}(M)$ stands for the trace of a matrix M. The set of all primitive columns of length n, will be denoted by $\mathbb{Z}^{n\times}$, whereas the set of all matrices of height m and width n with entries in a ring R is denoted by $R^{(m,n)}$. If $U \in R^{(n,m)}$ and $M \in R^{(n,n)}$ for some positive integers m and n, we set $M[U] := U^{\mathrm{tr}} M U$.

We will denote the symplectic group by $\mathrm{Sp}_n(\mathbb{R}) \subseteq \mathbb{R}^{(2n,2n)}$ and its subgroup of matrices with integer valued entries by Γ_n.

For a matrix $M \in \mathbb{C}^{(n,n)}$ we will denote the upper left submatrix in $\mathbb{C}^{(n-1,n-1)}$ by M_1.

The lower case letter p is reserved for a prime number. Given $N \in \mathbb{N}$ let p^n be the maximal power of p in N. Then we define $\nu_p(N) = n$. The symbol ϕ is used for the Euler ϕ-function and μ denotes the Möbius function.

The trivial character of period $N \in \mathbb{N}$ is denoted by $\mathbb{1}_N$. The Gauß sum associated to a Dirichlet character of period N is denoted by $G_\chi = \sum_{\alpha(N)} \chi(\alpha) e^{2\pi\imath\frac{\alpha}{N}}$.

The L-function over $\mathbb{Q}$ associated to the Dirichlet character χ is denoted by $L(s,\chi)$.

The Siegel upper halfspace of degree n will be denoted by $\mathbb{H}_n$. We will denote the Siegel modular group by $\Gamma_n := \mathrm{Sp}_n(\mathbb{Z})$. Given $Z \in \mathbb{H}_n$ and $n > 1$ we will write $Z = \left(\begin{smallmatrix} Z_1 & w \\ w^{\mathrm{tr}} & z \end{smallmatrix}\right)$, where $Z_1 \in \mathbb{C}^{(n-1,n-1)}$, $w \in \mathbb{C}^{n-1}$, $z \in \mathbb{C}$ and $Z = X + \imath Y, Z_1 = X_1 + \imath Y_1$, $w = u + \imath v$, $z = x + iy$, where $X_1, Y_1 \in \mathbb{R}^{(n-1,n-1)}, u, v \in \mathbb{R}^{n-1}$ and $x, y \in \mathbb{R}$.

The vector space of all (not necessarily holomorphic) Siegel modular forms of weight k with respect to a character χ associated to a group $\Gamma' \subseteq \mathrm{Sp}_n(\mathbb{R})$ will be denoted by $[\Gamma', k, \chi]$. The vector space of all corresponding cusp forms is denoted by $[\Gamma', k, \chi]_0$.

The well known action of $\mathrm{Sp}_n(\mathbb{R})$ on the meromorphic functions $\mathbb{H}_n \to \mathbb{C}$ given by the action on the argument is denoted

$$f|_k M(Z) = f|M(Z) = \det(CZ+D)^k f(M\langle Z\rangle)$$

with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. We will omit the weight k if it is clear by the context.

The $\mathrm{Sp}_n(\mathbb{R})$ invariant measure on $\mathbb{H}_n$ is denoted by $d\nu_n = (\det Y)^{-n-1} dX dY$. The $\mathrm{Sp}_{n-1} \times \mathbb{R}^{n-1}$ invariant measure on $\mathbb{H}_{n-1} \times \mathbb{C}^{n-1}$ will be denoted by $d\nu_n^* = (\det Y_1)^{-n-1} dX_1 dY_1 du dv$.

Let $\mathcal{F}_n$ be the Siegel fundamental domain (c.f. [1]) and $\mathcal{F}_n^*$ the corresponding fundamental domain for Jacobi forms

$$\mathcal{F}_n^* = \Big\{ Z \in \mathbb{H}_n \,:\, Z_1 \in \mathcal{F}_{n-1}, v \in \Phi(\mathbb{R}^{n-1}/Y_1\mathbb{Z}^{n-1}), \\ , |u_j| \le \frac{1}{2} \text{ for } 1 \le j \le n-1,\, u_1 \ge 0 \Big\},$$

where $\Phi(\mathbb{R}^{n-1}/Y_1\mathbb{Z}^{n-1})$ is a fundamental parallelogram of the given quotient.

Let p be an arbitrary prime number. Given a subgroup Γ' of $\Gamma_{2,1}^{(p)*}(p,p)$ (cf. (3.7) on page 15) of finite index with $-I \in \Gamma'$, let $\mathcal{F}'$ be a fundamental domain with respect to the action of Γ' on $\mathbb{H}_n$. If the integral converges, we define the normalized Petersson scalar product by

$$\{f,\, g\}_{\Gamma'}^p := \frac{1}{[\Gamma_{2,1}^{(p)*}(p,p) \,:\, \Gamma']} \int\limits_{\mathcal{F}'} f(Z)\overline{g(Z)}(\det Y)^k d\nu_n,$$

where $f, g \in [\Gamma', k, \chi]$ for a character χ. We may write $\{f,\, g\} = \{f,\, g\}_{\Gamma'}^p$, if it is clear that an appropriate Γ' exists and p is given by the context. Furthermore if $M \in \mathrm{Sp}(\mathbb{Q})$ normalizes Γ' and F is a fundamental domain of Γ', so is $M\langle F\rangle$ and thus $\{f,\, g\}_{\Gamma'} = \{f|M,\, g|M\}_{\Gamma'}$.

We will use the analogue for Jacobi forms only in cases of the full modular group. So, let $\mathcal{F}_n^*$ be a fundamental domain with respect to the action of $\Gamma_{n-1} \ltimes \mathbb{Z}^{n-1}$ on $\mathbb{H}_{n-1} \times \mathbb{C}^{n-1}$ and write

$$\langle f,\, g\rangle := \int\limits_{\mathcal{F}_n^*} f(Z_1, w)\overline{g(Z_1, w)}(\det Y_1)^k e^{-4\pi m Y_1^{-1}[v]} d\nu_n^*,$$

if f and g are Jacobi forms with respect to the same index m and the integral converges.

Definition (1.1) (Generalized Epstein ζ-function)
The generalized Epstein ζ-function for a positive definite Matrix $P \in \mathbb{R}^{(2n,2n)}$ and $u, v \in \mathbb{R}^{2n}$ is defined by

$$\zeta(s,u,v,P) := \sum_{\substack{\lambda \in \mathbb{Z}^{2n} \\ \lambda + v \neq 0}} e^{2\pi\imath u^{\mathrm{tr}}\lambda} P[\lambda+v]^{-s} \tag{1}$$

for $\mathfrak{Re}(s) > n$. Its completion is given by

$$\zeta^*(s,u,v,P) := \pi^{-s}\Gamma(s)\zeta(s,u,v,P). \tag{2}$$

□

Theorem (1.2)
The series for $\zeta(s,u,v,P)$ converges absolutely and locally uniformly if $\mathfrak{Re}(s) > n$. Its completion ζ^* has an analytic continuation to the whole s-plan with no poles if $u \notin \mathbb{Z}^{2n}$ and a simple pole at $s = n$ with residuum $(\det P)^{-\frac{1}{2}}$ if $u \in \mathbb{Z}^{2n}$. Furthermore we have the functional equation

$$\zeta^*(n-s,u,v,P) = (\det P)^{-\frac{1}{2}}\zeta^*(s,-v,u,P^{-1}) \qquad \square$$

Proof
cf. Terras [8]. ■

To apply these results to Eisenstein series we consider the following notation.

Lemma (1.3)
For $Z = X + \imath Y \in \mathbb{H}_n$ let

$$P_Z := \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \left[\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} \right]. \tag{3}$$

Then P_Z is a real symplectic positve definite matrix and for $M = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ we have

$$P_{M\langle Z\rangle} = P_Z[M^{\mathrm{tr}}], \quad P_{J\langle Z\rangle} = P_Z^{-1} \tag{4}$$

and moreover

$$(\mathfrak{Im}(M\langle Z\rangle))^{-1} = P_Z \left[\begin{pmatrix} C^{\mathrm{tr}} \\ D^{\mathrm{tr}} \end{pmatrix} \right] \qquad \text{and}$$

$$\frac{\det \mathfrak{Im}(M\langle Z\rangle)_1}{\det \mathfrak{Im}(M\langle Z\rangle)} = ((\mathfrak{Im} M\langle Z\rangle)^{-1})_{n,n} = P_Z[\lambda],$$

where λ is the last column of M^{tr}. □

Proof
For the last equation apply Cramer's rule to

$$\mathfrak{Im}(M\langle Z\rangle)v \stackrel{!}{=} e_n,$$

where e_n is the n-th standart orthonormal basis' vector. ■

Theorem (1.4)
Given $M \in \Gamma_n$ and $Z \in \mathbb{H}_n$ with P_Z as in (1.3) one has

$$\zeta^*(s,0,v,P_{M\langle Z\rangle}) = \frac{1}{s-n} + \int_1^\infty \sum_{\substack{\lambda\in\mathbb{Z}^{2n}\\ \lambda+v\neq 0}} e^{-\pi\theta P_{M\langle Z\rangle}[\lambda+v]}\theta^{s-1}d\theta$$
$$+ \int_1^\infty \sum_{\lambda\in\mathbb{Z}\setminus\{0\}} e^{-\pi\theta P_{M\langle Z\rangle}[\lambda]+2\pi\imath v^{\mathrm{tr}}\lambda}\theta^{n-s-1}d\theta. \qquad \square$$

Proof
Cf. Siegel [7]. ■

§ 2 Statement of the result

First we want to present the statement we are going to prove.

Definition (2.1) (Rankin convolution)
Let p be prime and let $f, g \in [\Gamma_{n,1}(p,p), k, \mathbb{1}_1]_0$ be Siegel cusp forms with Fourier-Jacobi series $f(Z) = \sum_{m=1}^{\infty} f_m(Z_1, w) e^{2\pi\imath mz}$ and $g(Z) = \sum_{m=1}^{\infty} g_m(Z_1, w) e^{2\pi\imath mz}$. Then the convolution of f and g is defined to be

$$D_{f,g}(s) = \sum_{m=1}^{\infty} \langle f_m, g_m \rangle m^{-s}.$$

Let $N \in \mathbb{N}$ be a natural, χ a Dirichlet character of period N. The twisted Rankin convolution and its completion is defined to be

$$D_{f,g,\chi}(s) = \sum_{m=1}^{\infty} \chi(m) \langle f_m, g_m \rangle m^{-s}$$
$$\mathbb{D}_{f,g,\chi}(s) = \left(\frac{2\pi}{N}\right)^{-2s} \Gamma(s)\Gamma(s-k+n) L(2s-2k+2n, \chi^2) D_{f,g,\chi}(s). \qquad \square$$

Remark (2.2)
It is already known that the twisted Rankin convolution admits analytic continuation. Here, this is a corollary of (7.2) on page 53 and (6.4) on page 36. □

Corollary (2.3)
Let p be a prime, $f, g \in [\Gamma_{2,1}^{(p)*}(p,p), k, \mathbb{1}_1^{k-}]_0$ (cf. (3.7) on page 15 and (4.9) on page 28). Let χ be a primitive character of period N such that $p \equiv 1\ (N)$.

$$\mathbb{D}_{f,g,\chi}(s) = \frac{G_\chi^4}{N^2} p^{3(k-s-1)} (1+p^{-(k-s)})(1+p^{-(s-k+2)})^{-1} \mathbb{D}_{f,g,\overline{\chi}}(2k-2-s). \qquad \square$$

Proof
Let $\chi^{(2)}$ be a primitive character which induces χ^2 and let $L^{(2)}R^{(2)}$ be the minimal period of χ^2. Set $\nu := \frac{N}{L^{(2)}R^{(2)}}$.

Combining (7.2) on page 53 and (8.2) on page 57 we obtain by means of (6.9) on page 46 that

$$\begin{aligned}
&\frac{2}{i_N R^{(2)}}\left(\frac{\pi}{N^2}\right)^{k-n} p^{\frac{3}{2}(s-k+2)-1}(1+p^{-(s-k+2)})\mathbb{D}_{f,\,g,\,\chi}(s)\\
&=\frac{\mu(R^{(2)})}{R^{(2)}}\frac{\chi^{(2)}(R^{(2)})}{G_{\overline{\chi}^{(2)}}}\mu(R^{(2)})\overline{\chi}^{(2)}(R^{(2)})G_\chi^4 G_{\overline{\chi}^{(2)}} p^{\frac{3}{2}(k-s)-1}(1+p^{-(k-s)})\\
&\quad\cdot\frac{2}{i_N N^2}\left(\frac{\pi}{N^2}\right)^{k-2}\mathbb{D}_{f,g,\overline{\chi}}(2k-2-s)\\
\Rightarrow\quad &\mathbb{D}_{f,\,g,\,\chi}(s)\\
&=p^{3(k-s-1)}(1+p^{-(k-s)})(1+p^{-(s-k+2)})^{-1}\frac{G_\chi^4}{N^2}\mathbb{D}_{f,g,\overline{\chi}}(2k-2-s) \qquad \blacksquare
\end{aligned}$$

We want to apply this to the spinor L-function. Gritsenko gave a representation of this in terms of a Rankin convolution. Unfortunately he didn't make this representation explicit. Hence we indicate how to obtain it, using his notation. Given a Siegel eigenform f which is a cusp form such that the Fourier-Jacobi coefficients satisfy $f_1 \equiv 0$, $f_p \not\equiv 0$, by [2, theorem (4.8)] we have

$$\begin{aligned}
&L(2s-2k+4, \mathbb{1}_p) D_{\mathrm{Sym}^p f, F_{f_p}}\\
&= L(2s-2k+4, \mathbb{1}_p)\sum_{m\geq 1}\langle f^*_{mp},\, m^{2-k} f_p \Big|_k T_-^{(1)}(m)\rangle (mp)^{-s}\\
&= p^{-s} L(2s-2k+4, \mathbb{1}_p)\sum_{m\geq 1}\langle f^*_{mp}\Big|_k T_+^{(1)}(m)\ ,\ f_p\rangle m^{-s}\\
&= p^{-s}\left((p-p^{2k-2s-1}\langle f_p, f_p\rangle + ((-1)^k p^{3-s-k} + p^{2-2s})\langle f_p \Big|_k T_-^{(1)}(p) T_+^{(1)}(p)\ ,\ f_p\rangle\right) Z_f(s)\\
&= p^{-s}(1-p^{k-2-s})(p+(-1)^k p^{k-s})\langle f_p, f_p\rangle Z_f(s).
\end{aligned}$$

The completion of the spinor ζ-function is given by

$$\mathbb{Z}_f(s) = (2\pi)^{-s}\Gamma(s)\cdot\Gamma(s-k+2) Z_f(s).$$

Hence we have

$$\begin{aligned}
&\mathbb{D}_{\mathrm{Sym}^p f, F_{f_p}}(s)\\
&= (2\pi)^{-s}\Gamma(s)\Gamma(s-k-2)\zeta(2s-2k-4) D_{\mathrm{Sym}^p f, F_{f_p}}(s)\\
&= p^{-s}\frac{1-p^{k-2-s}}{1-p^{-2s+2k-4}}(p+(-1)^k p^{k-s})\langle f_p, f_p\rangle \mathbb{Z}_f(s)\\
&= \frac{p^{-s}(p+(-1)^k p^{k-s})}{1+p^{k-2-s}}\langle f_p, f_p\rangle \mathbb{Z}_f(s)
\end{aligned}$$

If $p \equiv 1\,(N)$, we denote the twist by $\mathbb{Z}_f^\chi$. Since here twisting is compatible with completion we deduce

$$\begin{aligned}
&\frac{p^{-s}(p+(-1)^k p^{k-s})}{1+p^{k-2-s}}\langle f_p, f_p\rangle \mathbb{Z}_f^\chi(s)\\
&= \frac{G_\chi^4}{N^2} p^{3(k-s-1)}(1+p^{-(k-s)})(1+p^{-(s-k+2)})^{-1}\\
&\quad\cdot \frac{p^{-(2k-2-s)}(p+(-1)^k p^{k-(2k-2-s)})}{1+p^{-(2k-2-s)+k-2}}\langle f_p, f_p\rangle \mathbb{Z}_f^{\overline{\chi}}(2k-2-s)\\
\Rightarrow &\mathbb{Z}_f^\chi(s)\\
&= \frac{G_\chi^4}{N^2}\frac{p^s(1+p^{k-2-s})}{p+(-1)^k p^{k-s}} p^{3(k-s-1)}(1+p^{-(k-s)})(1+p^{-(s-k+2)})^{-1}\\
&\quad\cdot \frac{p^{-(2k-2-s)}(p+(-1)^k p^{k-(2k-2-s)})}{1+p^{-(2k-2-s)+k-2}}\mathbb{Z}_f^{\overline{\chi}}(2k-2-s)\\
&= (-1)^k \frac{G_\chi^4}{N^2} p^{k-s-1}\frac{p+(-1)^k p^{k-(2k-2-s)}}{p^{k-s}+(-1)^k p}\mathbb{Z}_f^{\overline{\chi}}(2k-2-s)\\
&= (-1)^k \frac{G_\chi^4}{N^2}\mathbb{Z}_f^{\overline{\chi}}(2k-2-s).
\end{aligned}$$

Corollary (2.4)
Let $f \neq 0$ be a Siegel eigenform which is a cusp form. Then the spinor L-function associated to any primitive Dirichlet character χ with given conductor N satisfies the functional equation

$$\mathbb{Z}_f^\chi(s) = (-1)^k \frac{G_\chi^4}{N^2}\mathbb{Z}_f^{\overline{\chi}}(2k-2-s). \qquad \square$$

Proof
The functional equation was proven in [5] if $f_1 \not\equiv 0$. Hence suppose $f_1 \equiv 0$. By the preceding calculations it suffices to prove that for each $N \in \mathbb{N}$ there is a prime $p_N \equiv 1\ (N)$ such that $f_{p_N} \not\equiv 0$.

Fix N and f. We proceed by first showing that there is a non-vanishing Fourier coefficient of f whose integral but not necessarily even lattice index contains one vector v such that the associated quadratic form evaluated at v - which we will call its value - is a unit and a square mod N. Then we will deduce that we may choose such a lattice to be primitive. Finally we use the correspondence of values mod N represented by quadratic forms and ray classes of quadratic number fields to obtain p_N.

Suppose there is no integral lattice with non-vanishing Fourier coefficient which contains one such vector v. Let Λ be any primitive integral lattice. We will show that is Fourier coefficient vanishes, too. This follows immediately if there is a vector whose value is a unit and a square mod N. Otherwise consider Theorem (6.1) in [4]. According to [10] Λ contains infinitely many vectors whose value is a prime. One of them has value which is a unit mod N. Choose a prime $q \nmid N$ which is non-square mod N. Using Wallings notation Λ^q contains therefor a vector whose value is a unit and a square mod N and hence all Fourier coefficients associated to any lattice containing Λ^q vanish. Since f is an eigenform so does the Fourier coefficient associated to Λ. The lattice Λ was arbitrary primitive and hence this is a contradiction to Lemma (4.7) in [3].

There is an integral quadratic form which represents a squared unit mod N and whose Fourier coefficient doesn't vanish. We claim that there is a such a quadratic form T which is primitive. We may chose a minimal value r, which is a square and a unit mod N, represented by any such form and consider the associated Fourier-Jacobi coefficient. Consider the proof of Theorem (4.7) in [3]. Using Gritsenko's notation the lines

$$f_d\{F|_k T(e)\}(\tau, z) = (f_r\{F\}|_k T_+(e))(\tau, z) \quad \text{and}$$
$$f_r(\tau, z)\exp(2\pi\imath r\omega) = \sum_{N=\left(\begin{smallmatrix} * & * \\ * & r \end{smallmatrix}\right)\in\mathfrak{B}_2} a(N)\exp(2\pi\imath(\mathrm{tr}(NZ)))$$

only involve quadratic forms which represent a squared unit mod N. To see this in the first line note that $\frac{r}{e}\frac{a}{b}$ is a unit and a square mod N if $ab = e \mid r$. Hence the argument of Gritsenko yields the claimed existence of T.

Note that T also represents a value $a_N \equiv 1$ (N). We now switch to a language developed by Zagier in [10]. The matrix T corresponds to an ideal class in the narrow sense $\mathfrak{k}$ of an imaginary quadratic field $K = Q(\sqrt{-\det(T)})$. The existence of a_N yields an element of $\mathfrak{k}$ which has norm a_N over $\mathbb{Q}$. We now consider the ray class group associated to the principle ideal $(N) \subseteq \mathfrak{O}_K$ in the integers of K. We know that there is ray class $\mathfrak{k}'$ containing an element which has norm $a_N \equiv 1$ (N). Then by theorem VII(13.2) of [6] there are infinitely many splitted prime ideals in $\mathfrak{k}'$ by the same argument Zagier uses in [10, §6]. This yields the existence of p_N. ■

§ 3 Subgroups and their systems of representatives

We first introduce some notation.

If $M \in \mathrm{Sp}_n(\mathbb{R})$ is arbitrary we write $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A, B, C, D \in \mathbb{R}^{(n,n)}$. The matrices A, B, C, D are written $A = \left(\begin{smallmatrix} A_1 & a_1 \\ a_2^{\mathrm{tr}} & \alpha \end{smallmatrix}\right)$ etc.. Here $A_1 \in \mathbb{R}^{(n-1,n-1)}, a_1, a_2 \in \mathbb{R}^{n-1}$ and $\alpha \in \mathbb{R}$. We denote $\tilde{M}, \hat{M}$ analogously.

Given $N\mathbb{N}$ we define three subgroups of Γ_n.

$$\Gamma_{n,1} := \{M \in \Gamma_n \,:\, c_2 = d_2 = 0,\, \gamma = 0,\, \delta = 1\}$$
$$\Gamma_{n,1}\left(N, \frac{N^2}{\kappa}\right) := \{M \in \Gamma_n \,:\, c_2 \equiv d_2 \equiv 0\ (N),\, \gamma \equiv 0\ \left(\frac{N^2}{\kappa}\right)\}$$
$$\Gamma^1_{n,1}\left(N, \frac{N^2}{\kappa}\right) := \{M \in \Gamma_{n,1}\left(N, \frac{N^2}{\kappa}\right) \,:\, \delta \equiv \pm 1\ (N)\}$$

It is well known that $\Gamma_{n,1}$ and $\Gamma_{n,1}\left(N, \frac{N^2}{\kappa}\right)$ are subgroups of Γ_n. To see that $\Gamma^1_{n,1}\left(N, \frac{N^2}{\kappa}\right)$ is a subgroup, too, observe that $\alpha \equiv \delta\ (N)$.

Set

$$\mathcal{F}_{n,1} = \left\{Z \in \mathbb{H}_n \,:\, (Z_1, w) \in \mathcal{F}^*_n,\, x \in [0,1],\, y > Y_1^{-1}[v]\right\}.$$

This is a fundamental domain with respect to the action of $\Gamma_{n,1}$ on $\mathbb{H}_n$ given for example by Kuß[5].

We define

$$W_\eta := \left(\begin{array}{cc|cc} & & I & \\ & & & \frac{1}{\eta} \\ \hline -I & & & \\ & -\eta & & \end{array}\right) \in \mathrm{Sp}_n(\mathbb{R}), \qquad M_\eta := \left(\begin{array}{cc|cc} I & & & \\ & 1 & & \eta \\ \hline & & I & \\ & & & 1 \end{array}\right) \in \mathrm{Sp}_n(\mathbb{R}),$$

$$D_\eta := \left(\begin{array}{cc|cc} I & & & \\ & \eta & & \\ \hline & & I & \\ & & & \frac{1}{\eta} \end{array}\right) \in \mathrm{Sp}_n(\mathbb{R}),$$

$$P_{d,t} := \left(\begin{array}{ccc|ccc} I & & & & & \\ & x\sqrt{d} & -\frac{t}{\sqrt{d}} & & & \\ & -\frac{y}{\sqrt{d}} & \sqrt{d} & & & \\ \hline & & & I & & \\ & & & & \sqrt{d} & \frac{y}{\sqrt{d}} \\ & & & & \frac{t}{\sqrt{d}} & x\sqrt{d} \end{array}\right) \in \mathrm{Sp}_n(\mathbb{R}),$$

where $d \mid t$, $(d, \frac{t}{d}) = 1$, $xd - y\frac{t}{d} = 1$ and $\eta \in \mathbb{R}$. We set $J := W_1$.

Given $\lambda \in \mathbb{Z}^{2(n-1)\times}$ let

$$M_\lambda := \left(\begin{array}{cc|cc} A_\lambda & & B_\lambda & \\ & 1 & & \\ \hline C_\lambda & & D_\lambda & \\ & & & 1 \end{array}\right),$$

where $\left(\begin{smallmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{smallmatrix}\right) \in \Gamma_{n-1}$ is an abritray but fixed matrix which has last row λ^{tr}.

We consider a subgroup of $\mathrm{Sp}(\mathbb{Q})$ with non integral entries.

Definition (3.1)

Let $N, \tau, t \in \mathbb{N}$ be natural numbers where t is square free and $t \mid \tau$, $(N, \tau) = 1$. Furthermore suppose $p \mid \tau \Rightarrow p \mid t$. If $\kappa \mid N\tau$, we define

$$\Gamma_{n,1}^{(t)}\left(N\tau, \frac{(N\tau)^2}{\kappa}\right) = \left\langle \Gamma_{n,1}\left(N\tau, \frac{(N\tau)^2}{\kappa}\right) \cup \left\{ M_{\frac{1}{p}} : p \mid t \right\} \right\rangle \leq \mathrm{Sp}_n(\mathbb{Q}).$$

Analogously define

$$\Gamma_{n,1}^{(t)} = \left\langle \Gamma_{n,1} \cup \left\{ M_{\frac{1}{p}} : p \mid t \right\} \right\rangle \leq \mathrm{Sp}_n(\mathbb{Q}).$$

A fundamental domain for $\Gamma_{n,1}^{(t)}$ is

$$\mathcal{F}_{n,1}^{(t)} := \{ Z \in \mathcal{F}_{n,1} \,:\, x \in [0, \frac{1}{t}] \}. \qquad \square$$

Proposition (3.2)

We have

$$\Gamma_{n,1}^{(t)}\left(N\tau, \frac{(N\tau)^2}{\kappa}\right) \subseteq \{ M \in \mathrm{Sp}_n(\mathbb{Q}) \,:\, A,\, C,\, D,\, B_1,\, b_1,\, b_2,\, t \cdot \beta \text{ integral},$$
$$a_1,\, c_1,\, c_2,\, d_2 \equiv 0\ (N\tau),\ \gamma \equiv 0\ \left(\frac{(N\tau)^2}{\kappa}\right) \Big\}.$$

Moreover, if $p \mid t$ is a prime and $p^2 \mid \tau \vee p \nmid \kappa$ we have $(p,\delta)=1$. Otherwise given $p \mid t$ the last row satisfies either $(p,\delta)=1$ or $p \mid \delta \wedge p^2 \nmid \gamma$.

Conversely, each matrix satisfying all these statements is in $\Gamma_{n,1}^{(t)}(N\tau, \frac{(N\tau)^2}{\kappa})$. □

Proof

Since we extend $\Gamma_{n,1}\left(N\tau, \frac{(N\tau)^2}{\kappa}\right)$ by the matrices $M_{\frac{1}{p}}$ it suffices to check whether the given set is stable under left and right multiplication by all these matrices, which is immediate.

For the converse statement suppose M is given satisfying all properties. If $p \mid \delta$ and $p \mid t$ is prime, we may multiply by $M_{\frac{1}{p}}$ from the right and hence assume that $(p,\delta)=1$. Thus we may assume $(t,\delta)=1$.

Now choose $\eta \in \mathbb{Z}$ such that $\eta\delta \equiv t\beta\ (t)$. Then $M_{-\frac{\eta}{t}}M$ has integral entries and thus is an element of $\Gamma_{n,1}(N\tau, \frac{(N\tau)^2}{\kappa})$, which proofs the last statement. ■

We state some facts for reference, sketching the well known proofs.

Lemma (3.3)

Let t be square free. Given $M \in \Gamma_{n,1}^{(t)}(Nt, N^2t)$ we have $W_{Nt}MW_{Nt}^{-1} \in \Gamma_{n,1}^{(t)}(Nt, N^2t)$ with lower right entry α. □

Proof

Let $\tilde{D}_\eta = \operatorname{diag}(1,\ldots,1,\eta) \in \mathbb{R}^{(n,n)}$ for $\eta \in \mathbb{Q}$. Then we have

$$W_{Nt}MW_{Nt}^{-1} = \begin{pmatrix} \tilde{D}_{\frac{1}{Nt}} D \tilde{D}_{Nt} & -\tilde{D}_{\frac{1}{Nt}} C \tilde{D}_{\frac{1}{Nt}} \\ -\tilde{D}_{Nt} B \tilde{D}_{Nt} & \tilde{D}_{Nt} A \tilde{D}_{\frac{1}{Nt}} \end{pmatrix}.$$

Using the properties for $A, B, C, D \bmod Nt$ the statement follows. ■

Lemma (3.4)

The matrix M_N^{tr} normalises $\Gamma_{n,1}^1(N, N^2)$. □

Proof

Given $M \in \Gamma_{n,1}^1(N,N^2)$, the only problem is to consider $(M_N^{\mathrm{tr}} M M_N^{-\mathrm{tr}})_{2n,n}$. This entry yields $\gamma + N\alpha - N\delta - N^2\beta$. Since $\alpha \equiv \delta\ (N)$ the conjugated matrix is in $M \in \Gamma_{n,1}^1(N,N^2)$, too. ■

—— $\Gamma_{2,1}^{(p)*}(Np, \frac{N^2}{\kappa}p)$ ——

We define a further extension of the treated groups. This is a generalization of the normal extension of $\Gamma_{2,1}^{(p)}(p,p)$ given by Gritsenko [3].

Definition (3.5)
Let p be a prime and suppose that $N \in \mathbb{N}$ such that $(N, p) = 1$. Furthermore suppose that $\kappa \mid N$. Let $p^* \in \mathbb{N}$ be arbitrary such that $p^*p \equiv 1\ (N)$. Choose x, $y \in \mathbb{Z}$ such that $xp - y = 1$, e.g. $x = 1$, $y = p - 1$. Set

$$\begin{aligned} H_p(N) &:= \begin{pmatrix} U^{-\mathrm{tr}} & \\ & U \end{pmatrix} P_{p,p} \\ &= \frac{1}{\sqrt{p}} \left(\begin{array}{cc|cc} p(x-p^*y) & -p(1-p^*p) & & \\ 1-p^*py & p^*p^2 & & \\ \hline & & p^*p^2 & -1+p^*py \\ & & p(1-p^*p) & p(x-p^*y) \end{array}\right), \text{ where} \\ U &= \begin{pmatrix} 1+p^*p & -1 \\ -p^*p & 1 \end{pmatrix}. \end{aligned}$$

□

Remark (3.6)
One may ask why this definition makes sense. The matrix $\left(\begin{smallmatrix} U^{-\mathrm{tr}} & \\ & U \end{smallmatrix}\right)$ is in $\Gamma_{2,1}^{(p)}(p, p)$ and $P_{p,p}$ usually is used to define a normal extension of this group. Since $N \mid (\sqrt{p} \cdot H_p(N))_{4,3}$ we may use $H_p(N)$ to define an extension of $\Gamma_{2,1}^{(p)}(Np, \frac{N^2}{\kappa}p)$. □

Definition (3.7)
Let p be a prime and suppose $N \in \mathbb{N}$ such that $(N, p) = 1$. Furthermore suppose $\kappa \mid N$. We set

$$\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right) := \left\langle \Gamma_{2,1}^{(p)}\left(Np, \frac{N^2}{\kappa}p\right) \cup \{H_p(N)\} \right\rangle .$$

Furthermore we define

$$i_N := [\Gamma_{2,1}^{(p)*}(p, p) \,:\, \Gamma_{2,1}^{(p)*}(Np, N^2p)].$$

□

This clearly is a normal extension of $\Gamma_{2,1}^{(p)}\left(Np, \frac{N^2}{\kappa}p\right)$ of index 2. We want to consider the last rows of matrices in this group.

Lemma (3.8)
Let p be a prime and suppose $N \in \mathbb{N}$ such that $(N, p) = 1$. Furthermore suppose $\kappa \mid N$. Let $\lambda \in \mathbb{Z}^{4\times}$ run through all primitive columns with $(\lambda_4, N) = 1$. Then the

last row of any matrix in $\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right)$ is as follows.

$$\left(pN\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad pN\lambda_3 \quad \lambda_4\right), \quad \text{where } p \nmid \lambda_4,$$
$$\left(pN\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad pN\lambda_3 \quad p\lambda_4\right), \quad \text{where } p \nmid \lambda_2,$$
$$\sqrt{p}\left(N\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad N\lambda_3 \quad \lambda_4\right), \quad \text{where } p \nmid \lambda_1,$$
$$\sqrt{p}\left(pN\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad N\lambda_3 \quad \lambda_4\right), \quad \text{where } p \nmid \lambda_3.$$

For each such row r there is a matrix in $\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right)$ with last row r. □

Proof

The first and second types of matrix have already been discussed in (3.2). Since $H_p(N)$ generates the extension of index 2 it suffices to multiply these rows by $H_p(N)$ or $H_p(N)^{-1}$. Hence let $p^* \in N$ such that $p^*p \equiv 1\ (N)$. Suppose $p \nmid \lambda_4$ and consider

$$\begin{aligned}&\left(pN\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad pN\lambda_3 \quad \lambda_4\right) H_p(N)\\ &= \sqrt{p}\Big(N(p(x-p^*y)\lambda_1 + \tfrac{N}{\kappa}(1-p^*py)\lambda_2) \quad \tfrac{N^2}{\kappa}(-p\kappa\tfrac{1-p^*p}{N}\lambda_1 + p^*p^2\lambda_2)\\ &\qquad N(p^*p^2\lambda_3 + \tfrac{1-p^*p}{N}\lambda_4) \quad N(-1+p^*py)\lambda_3 + (x-p^*y)\lambda_4\Big).\end{aligned}$$

Since $x - p^*y \equiv p^*\ (N)$ we see that the last entry and N are coprime. Suppose $p \nmid \lambda_2$. Then the first entry and p are coprime and the row is of the type

$$\sqrt{p}\left(N\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad N\lambda_3 \quad \lambda_4\right), \quad \text{where } p \nmid \lambda_1.$$

Otherwise consider $(p, 1-pp^*) = 1$ and hence the row is of the type

$$\sqrt{p}\left(pN\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad N\lambda_3 \quad \lambda_4\right), \quad \text{where } p \nmid \lambda_3.$$

Next we suppose $p \nmid \lambda_2$ and consider

$$\begin{aligned}&\left(pN\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad pN\lambda_3 \quad p\lambda_4\right) H_p(N)\\ &= \sqrt{p}\Big(N(p(x-p^*y)\lambda_1 + \tfrac{N}{\kappa}(1-p^*py)\lambda_2) \quad \tfrac{N^2}{\kappa}(-p\kappa\tfrac{1-p^*p}{N}\lambda_1 + p^*p^2\lambda_2)\\ &\qquad N(p^*p^2\lambda_3 + p\tfrac{1-p^*p}{N}\lambda_4) \quad N(-1+p^*py)\lambda_3 + p(x-p^*y)\lambda_4\Big).\end{aligned}$$

This always is of the type

$$\sqrt{p}\left(N\lambda_1 \quad p\tfrac{N^2}{\kappa}\lambda_2 \quad N\lambda_3 \quad \lambda_4\right), \quad \text{where } p \nmid \lambda_1.$$

We now show, that each last row occurs as a last row of a matrix in $\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right)$. To do so, we multiply by $H_p(N)^{-1}$. Then the result should be in $\Gamma_{2,1}^{(p)}\left(Np, \frac{N^2}{\kappa}p\right)$ and if the last row is valid according to (3.2) on page 13, we are done.

So, suppose $p \nmid \lambda_1$.

$$\begin{aligned}
&\sqrt{p}\begin{pmatrix} N\lambda_1 & p\frac{N^2}{\kappa}\lambda_2 & N\lambda_3 & \lambda_4 \end{pmatrix} H_p(N)^{-1} \\
&= \Big(N(p^*p^2 - \tfrac{N}{\kappa}p(-1+p^*py)\lambda_2 \quad \tfrac{N^2}{\kappa}(p\kappa\tfrac{1-p^*p}{N}\lambda_1 + p^2(x-p^*y)\lambda_2) \\
&\qquad N(p(x-p^*y)\lambda_3 - p\tfrac{1-p^*p}{N}\lambda_4) \quad N(1-p^*py)\lambda_3 + p^*p^2\lambda_4 \Big).
\end{aligned}$$

If $p \nmid \lambda_3$ this is of the type

$$\begin{pmatrix} pN\lambda_1 & p\frac{N^2}{\kappa}\lambda_2 & pN\lambda_3 & \lambda_4 \end{pmatrix}, \quad \text{where } p \nmid \lambda_4.$$

Otherwise it is of the type

$$\begin{pmatrix} pN\lambda_1 & p\frac{N^2}{\kappa}\lambda_2 & pN\lambda_3 & p\lambda_4 \end{pmatrix}, \quad \text{where } p \nmid \lambda_2.$$

Finally suppose $p \nmid \lambda_3$ and consider

$$\begin{aligned}
&\sqrt{p}\begin{pmatrix} pN\lambda_1 & p\frac{N^2}{\kappa}\lambda_2 & N\lambda_3 & \lambda_4 \end{pmatrix} H_p(N)^{-1} \\
&= \Big(N(p^*p^3 - \tfrac{N}{\kappa}p(-1+p^*py)\lambda_2 \quad \tfrac{N^2}{\kappa}(p^2\kappa\tfrac{1-p^*p}{N}\lambda_1 + p^2(x-p^*y)\lambda_2) \\
&\qquad N(p(x-p^*y)\lambda_3 - p\tfrac{1-p^*p}{N}\lambda_4) \quad N(1-p^*py)\lambda_3 + p^*p^2\lambda_4 \Big).
\end{aligned}$$

This always is of the type

$$\begin{pmatrix} pN\lambda_1 & p\frac{N^2}{\kappa}\lambda_2 & pN\lambda_3 & \lambda_4 \end{pmatrix}, \quad \text{where } p \nmid \lambda_4. \qquad \blacksquare$$

Corollary (3.9)

Let p be a prime and suppose $N \in \mathbb{N}$ such that $(N,p) = 1$. Furthermore suppose $\kappa \mid N$. Then a system of representatives of $\Gamma_{2,1}^{(p)} \backslash \Gamma_{2,1}^{(p)*}(Np, \frac{N^2}{\kappa}p)$ is given by a set of matrices M_r with last rows r as given in (3.8). □

Proof

Given a matrix $M \in \Gamma_{2,1}^{(p)*}(Np, \frac{N^2}{\kappa}p)$ with last row r, the matrix $M_r M^{-1}$ has last row $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ and hence $M_r M^{-1} \in \Gamma_{2,1}^{(p)}$.

Analogously, given two rows r and r' we have $M_r M_{r'}^{-1} \in \Gamma_{2,1}^{(p)}$ if and only if $r = r'$. $\blacksquare$

—— *Tranvsersals of* $\Gamma_{n,1}\left(Np, \frac{N^2}{\theta}p\right)$ ——

Lemma (3.10)
Let $N, p \in \mathbb{N}$ be coprime naturals and let $\nu, \theta \mid N$. Then a set of representatives of $\Gamma_{n,1}(N\nu p, (N\nu)^2 p)\backslash\Gamma_{n,1}\left(Np, \frac{N^2}{\theta}p\right)$ is given by matrices with last row

$$\begin{pmatrix} Npc_2^{\mathrm{tr}} & \frac{N^2}{\theta}p\gamma & Npd_2^{\mathrm{tr}} & 1 \end{pmatrix},$$

where c_2, d_2 $((\mathbb{Z}/\nu\mathbb{Z}))^{n-1}$ and $\gamma(\mathbb{Z}/\theta\nu^2\mathbb{Z})$ run through sets of representatives. □

Proof
Let $M_{c_2,\gamma,d_2,\delta}$ denote an arbitrary but fixed matrix in Γ_n with last row $\left(\begin{smallmatrix} c_2^{\mathrm{tr}} & \gamma & d_2^{\mathrm{tr}} & \delta \end{smallmatrix} \right)$.

Suppose $\widetilde{M} \in \Gamma_{n,1}\left(Np, \frac{N^2}{\theta}p\right)$. Since $(\tilde{\delta}, Np) = 1$, we see $(\tilde{\delta}, N\nu p) = 1$. And hence $\left(\begin{smallmatrix} N\nu p\tilde{b}_1^{\mathrm{tr}} & (N\nu)^2 p\tilde{\beta} & N\nu p\tilde{d}_1^{\mathrm{tr}} & \tilde{\delta} \end{smallmatrix} \right)$ is primitive. Thus we find $\hat{c}_2, \hat{d}_2 \in \mathbb{Z}^{n-1}, \hat{\gamma}, \hat{\delta} \in \mathbb{Z}$. such that

$$\begin{aligned} 1 &= \begin{pmatrix} \hat{c}_2^{\mathrm{tr}} & \hat{\gamma} & \hat{d}_2^{\mathrm{tr}} & \hat{\delta} \end{pmatrix} \begin{pmatrix} N\nu p\tilde{b}_1^{\mathrm{tr}} & (N\nu)^2 p\tilde{\beta} & N\nu p\tilde{d}_1^{\mathrm{tr}} & \tilde{\delta} \end{pmatrix}^{\mathrm{tr}} \\ &= \begin{pmatrix} N\nu p\hat{c}_2^{\mathrm{tr}} & (N\nu)^2 p\hat{\gamma} & N\nu p\hat{d}_2^{\mathrm{tr}} & \hat{\delta} \end{pmatrix} \begin{pmatrix} \tilde{b}_1^{\mathrm{tr}} & \tilde{\beta} & \tilde{d}_1^{\mathrm{tr}} & \tilde{\delta} \end{pmatrix}^{\mathrm{tr}} \end{aligned}$$

Hence there is a matrix in $\Gamma_{n,1}(N\nu p, (N\nu)^2 p)$ with last row $\left(\begin{smallmatrix} N\nu p\hat{c}_2^{\mathrm{tr}} & (N\nu)^2 p\hat{\gamma} & N\nu p\hat{d}_2^{\mathrm{tr}} & \hat{\delta} \end{smallmatrix} \right)$. Left multiplication by this matrix allows to assume $\tilde{\delta} = 1$.

In our sets of representatives for c_2, d_2 and γ we may choose representatives such that

$$\begin{aligned} &Npc_2 \equiv \tilde{c}_2 \ (N\nu p) \qquad Npd_2 \equiv \tilde{d}_2 \ (N\nu p) \\ &\frac{N^2}{\theta}p\gamma \equiv \tilde{\gamma} + (\tilde{c}_2 - Npc_2)^{\mathrm{tr}}\tilde{d}_2 \\ &\qquad\qquad - (\tilde{d}_2 - Npd_2)^{\mathrm{tr}}\tilde{c}_2 \quad ((N\nu)^2 p)\,. \end{aligned}$$

Then

$$\begin{pmatrix} Npc_2^{\mathrm{tr}} & \frac{N^2}{\theta}p\gamma & Npd_2^{\mathrm{tr}} & 1 \end{pmatrix} \widetilde{M}^{-1} \equiv \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix} \ (N\nu p)$$

since the given row is the last row of $\tilde{M} \bmod (N\nu p)$.

Consider

$$\begin{aligned} &\left(\begin{pmatrix} Npc_2^{\mathrm{tr}} & \frac{N^2}{\theta}p\gamma & Npd_2^{\mathrm{tr}} & 1 \end{pmatrix} \widetilde{M}^{-1} \right)_{2n,n} \\ &= \begin{pmatrix} Npc_2^{\mathrm{tr}} & \frac{N^2}{\theta}p\gamma & Npd_2^{\mathrm{tr}} & 1 \end{pmatrix} \begin{pmatrix} \tilde{d}_2^{\mathrm{tr}} & 1 & -\tilde{c}_2^{\mathrm{tr}} & -\tilde{\gamma} \end{pmatrix}^{\mathrm{tr}} \\ &\equiv 0 \quad ((N\nu)^2 p). \end{aligned}$$

Putting all together we have $M_{Npc_2,\frac{N^2}{\theta}p\gamma,Npd_2,1}\widetilde{M}^{-1} \in \Gamma_{n,1}(N\nu p,(N\nu)^2p)$.

Hence there remains to prove that the given matrices belong to distinct cosets. Therefore suppose that there is $M \in \Gamma_{n,1}(N\nu p,(N\nu)^2p)$ such that

$$MM_{Npc_2,\frac{N^2}{\theta}p\gamma,Npd_2,1} = M_{Np\tilde{c}_2,\frac{N^2}{\theta}p\tilde{\gamma},Np\tilde{d}_2,1}.$$

Since $(MM_{Npc_2,\frac{N^2}{\theta}p\gamma,Npd_2,1})_{2n,2n} = 1$ we see $M_{2n,2n} \equiv 1\ (N\nu p)$. Hence

$$M_{2n,*} = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}\ (N\nu p)$$

and $\tilde{c}_2 = c_2$, $\tilde{d}_2 = d_2$. Regarding

$$\frac{N^2}{\theta}p(\gamma-\tilde{\gamma}) = \left(M_{Npc_2,\frac{N^2}{\theta}p\gamma,Npd_2,1}M^{-1}_{Np\tilde{c}_2,\frac{N^2}{\theta}p\tilde{\gamma},Np\tilde{d}_2,1}\right)_{2n,n} \equiv 0\ ((N\nu)^2p)$$

we see $\tilde{\gamma} = \gamma$. ■

Corollary (3.11)
Let $N, p \in \mathbb{N}$ be coprime naturals and let $\nu, \theta \mid N$. Let $\mathfrak{e} := \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}^{\mathrm{tr}} \in \mathbb{Z}^{n-1}$ and define

$$M_{d,\gamma} := \left(\begin{array}{cc|cc} I & -Ndp\mathfrak{e} & & \\ & 1 & & \\ \hline & & I & \\ & \frac{N^2}{\theta}p\gamma & Ndp\mathfrak{e}^{\mathrm{tr}} & 1 \end{array}\right).$$

Then

$$\left\{M_{d,\gamma}M_\lambda\ :\ \gamma \in \mathbb{Z}/\theta\nu^2\mathbb{Z},\ d \mid \nu,\ \lambda \in \left\{1,\dots,\frac{\nu}{d}\right\}^{2(n-1)\times}\right\}$$

with M_λ as on page 3 is a set of representatives of $\Gamma_{n,1}(N\nu p,(N\nu)^2p)\backslash\Gamma_{n,1}\left(Np, \frac{N^2}{\theta}p\right)$.□

Proof
The last rows of $M_{d,\gamma}M_\lambda$ run through all rows given in (3.10). ■

Corollary (3.12)
Let $N, p \in \mathbb{N}$ be coprime naturals and let $\nu, \theta \mid N$. A system of representatives of $\Gamma^{(p)*}_{2,1}(N\nu p,(N\nu)^2p)\backslash\Gamma^{(p)*}_{2,1}\left(Np, \frac{N^2}{\theta}p\right)$ is given by the matrices of (3.11) with $n = 2$.□

Proof
Suppose $M \in \Gamma^{(p)*}_{2,1}(Np, \frac{N^2}{\theta}p)$. If M has non rational entries we may multiply by $H_p(N\nu)$. Hence we may assume $M \in \Gamma^{(p)}_{2,1}(Np, \frac{N^2}{\theta}p)$.

If $\delta \in p\mathbb{Z}$ we see $\beta \in \frac{1}{p}\mathbb{Z} \setminus \mathbb{Z}$ and hence by multiplication by $M^{\mathrm{tr}}_{(N\nu)^2 p}$ from the left we may assume $\delta \notin p\mathbb{Z}$. Then if $\beta \notin \mathbb{Z}$ multiplication by $M_{\frac{\eta}{p}}$ from the left, η chosen appropriately, yields $M \in \Gamma_{2,1}(Np, N\nu p)$ and we find an appropriate representative in (3.11).

We have to show that these representatives belong to distinct cosets. But this is immediate since given $M_{d,\gamma}M_\lambda$, $M_{\tilde{d},\tilde{\gamma}}M_{\tilde{\lambda}} \in \Gamma_{2,1}(Np, N\nu p)$ we have $M_{d,\gamma}M_\lambda(M_{\tilde{d},\tilde{\gamma}}M_{\tilde{\lambda}})^{-1} \in \Gamma^{(p)*}_{2,1}(N\nu p, (N\nu)^2 p)$ if and only if $M_{d,\gamma}M_\lambda(M_{\tilde{d},\tilde{\gamma}}M_{\tilde{\lambda}})^{-1} \in \Gamma_{2,1}(N\nu p, (N\nu)^2 p)$. ∎

§ 4 Dirichlet characters and their sums

Most statements of this section are due to Kuß [5]. The extension of a character of $\Gamma_{2,1}^{(p)}(p,p)$ to a character of $\Gamma_{2,1}^{(p)*}(p,p)$ is due the author.

—— *General properties* ——

The following lemmas are used to establish the functional equation of the Eisenstein series.

Lemma (4.1)
Let $N, L \in \mathbb{N}$ such that $L \mid N$. Let χ_L be a primitive Dirichlet character of conductor L and let χ be its induced character with period N. Then the minimal period of χ equals LR, where

$$R = \prod_{\substack{p \mid N \\ p \nmid L}} p.$$

In particular R is square free and $(L,R) = 1$. □

Proof
Considered as functions $\mathbb{Z} \to \mathbb{C}$ we have $\chi = \mathbb{1}_N \chi_L$ and the minimal period of $\mathbb{1}_N$ is $\prod_{p \mid N} p$. Since χ_L is primitive its minimal period is L. Hence the minimal period of $\mathbb{1}_N \chi_L$ which is the least common multiple of these periods equals

$$L \prod_{\substack{p \mid N \\ p \nmid L}} p = LR. \qquad \blacksquare$$

Lemma (4.2)
Let χ_L be a primitive Dirichlet character of period L and ρ a positive square free number such that $(L,\rho) = 1$. Then if $m \in \mathbb{Z}$ we have

$$\sum_{\delta (L\rho)} \left(\mathbb{1}_\rho \chi_L\right)(\delta) e^{2\pi\imath m \frac{\delta}{L\rho}} = \chi_L(\rho)\overline{\chi_L}(m)\mu(r)\phi\left(\frac{\rho}{r}\right) G_{\chi_L},$$

where $r := \frac{\rho}{(\rho, m)}$ and μ is the Möbius function. □

Proof
Since $(L,\, \rho) = 1$ we have

$$\sum_{\delta(L\rho)} \left(\mathbb{1}_\rho \chi_L\right)(\delta) e^{2\pi\imath m \frac{\delta}{L\rho}} = \sum_{\alpha(L),\, \beta(\rho)} \left(\mathbb{1}_\rho \chi_L\right)(\alpha\rho + \beta L) e^{2\pi\imath m \left(\frac{\alpha}{L} + \frac{\beta}{\rho}\right)}$$
$$= \chi_L(\rho) \sum_{\alpha(L)} \chi_L(\alpha) e^{2\pi\imath m \frac{\alpha}{L}} \sum_{\beta(\rho)} \mathbb{1}_\rho(\beta) e^{2\pi\imath m \frac{\beta}{\rho}}.$$

Since χ_L is primitive, we have

$$\chi_L(\rho) \sum_{\alpha(L)} \chi_L(\alpha) e^{2\pi\imath m \frac{\alpha}{L}} = \chi_L(\rho) \overline{\chi_L}(m) G_{\chi_L}.$$

We now consider the sum over β.

$$\sum_{\beta(\rho)} \mathbb{1}_\rho(\beta) e^{2\pi\imath m \frac{\beta}{\rho}} = \sum_{\beta(\rho)^\times} e^{2\pi\imath m \frac{\beta}{\rho}} = \sum_{\beta(\rho)} \left(\sum_{a \,|\, (\rho,\, \beta)} \mu(a) \right) e^{2\pi\imath m \frac{\beta}{\rho}}$$
$$= \sum_{a \,|\, \rho} \mu(a) \sum_{\substack{\beta(\rho) \\ a \,|\, \beta}} e^{2\pi\imath m \frac{\beta}{\rho}} = \sum_{a \,|\, \rho} \mu\left(\frac{\rho}{a}\right) \sum_{\beta(a)} e^{2\pi\imath m \frac{\beta}{a}}$$
$$= \sum_{a \,|\, \rho} \mu\left(\frac{\rho}{a}\right) \cdot \begin{cases} a & \text{, if } a \mid m \\ 0 & \text{, otherwise} \end{cases}$$
$$= \sum_{a \,|\, (\rho,\, m)} \mu\left(\frac{\rho}{a}\right) a = \sum_{a \,|\, (\rho,\, m)} \mu\left(\frac{a\rho}{(\rho,\, m)}\right) \frac{(\rho,\, m)}{a}.$$

By definition of r we have $(\rho,\, m) = \frac{\rho}{r}$ and because ρ is square free the last expression can be reduced to yield

$$\sum_{\beta(\rho)} \mathbb{1}_\rho(\beta) e^{2\pi\imath m \frac{\beta}{\rho}} = \sum_{a \,|\, \frac{\rho}{r}} \mu(ra) \frac{\rho}{ra} = \mu(r) \frac{\rho}{r} \sum_{a \,|\, \frac{\rho}{r}} \mu(a) \frac{1}{a}$$
$$= \mu(r) \frac{\rho}{r} \prod_{p \,|\, \frac{\rho}{r}} \left(\sum_{a \,|\, p} \mu(a) \frac{1}{a} \right) = \mu(r) \frac{\rho}{r} \prod_{p \,|\, \frac{\rho}{r}} \left(1 - \frac{1}{p}\right)$$
$$= \mu(r) \phi\left(\frac{\rho}{r}\right)$$

Putting all together we have

$$\sum_{\delta(L\rho)} \left(\mathbb{1}_\rho \chi_L\right)(\delta) e^{2\pi\imath m \frac{\delta}{L\rho}} = \chi_L(\rho) \overline{\chi_L}(m) \mu(r) \phi\left(\frac{\rho}{r}\right) G_{\chi_L}.$$

■

Lemma (4.3)
Let χ a primitive character with period N. Let $m \in \mathbb{N}$ be natural such that $N \nmid m$. Then there is $\rho \in \mathbb{Z}$ such that $(N, \rho) = 1$, $\rho \equiv 1\ (m)$ and $\chi(\rho) \neq 1$. □

Proof
Since $N \nmid m$, there is a prime p such that

$$\alpha := \nu_p(N) > \nu_p(m) =: \beta.$$

We have $\chi = \chi_p \cdot \widetilde{\chi}_p$ where χ_p is primitive with period p^α and $\widetilde{\chi}_p$ is primitive with period $\frac{N}{p^\alpha}$. Since $\left(p, \frac{Nm}{p^{\alpha+\beta}}\right) = 1$ there is $p^* \in \mathbb{Z}$ such that $pp^* \equiv 1 \left(\frac{Nm}{p^{\alpha+\beta}}\right)$.

We first consider χ_p. If $\alpha = 1$ let $r = p - 1$. Otherwise suppose for every $c \in \mathbb{Z}$ we have $\chi_p(cp^{\alpha-1} + 1) = 1$. Then for any $a \in \mathbb{Z}$, $(a, p) = 1$ chose $a^* \in \mathbb{Z}$ such that $aa^* \equiv 1\ (p)$ and regard $c = -a^*$. Then we have

$$\begin{aligned} \chi_p(a + p^{\alpha-1}) &= \chi_p(-a^* p^{\alpha-1} + 1)\chi_p(a + p^{\alpha-1}) \\ &= \chi_p\left(\left(-a^* p^{\alpha-2} - \frac{aa^* - 1}{p}\right) p^\alpha + a\right) \\ &= \chi_p(a). \end{aligned}$$

Hence χ_p has period $p^{\alpha-1}$ and this is a contradiction. So choose $c \in \mathbb{Z}$ such that $\chi_p(r) \neq 1$ with $r = cp^{\alpha-1} + 1$.

We now let $\rho := r + (pp^*)^\alpha(1 - r)$. Then we have $\rho \equiv r\ (p^\alpha)$ and hence $\rho \equiv 1\ (p^\beta)$. By choice of p^* we see $\rho \equiv 1 \left(\frac{Nm}{p^{\alpha+\beta}}\right)$. Putting all together we get $\rho \equiv 1\ (m)$. Since $\rho \equiv 1 \left(\frac{N}{p^\alpha}\right)$ we have $\chi(\rho) = \chi_p(r)\widetilde{\chi}_p(1) \neq 1$. ■

Corollary (4.4)
Let χ be a primitive character with period N. Suppose $\mu, r \in \mathbb{Z}$ and $p \mid N$, $(r, p) = 1$ for a prime p. Then we have

$$\sum_{\nu(p)} \chi(\mu + \frac{Nr}{p}\nu) = 0.$$ □

Proof
Let $\alpha := \nu_p(N)$. If $N \neq p^\alpha$ we have $\chi = \chi_p \cdot \widetilde{\chi}_p$ where χ_p is primitive with period p^α and $\widetilde{\chi}_p$ is primitive with period $\frac{N}{p^\alpha}$. Since $\widetilde{\chi}_p(\frac{Nr}{p}) = 0$ it suffices to consider $N = p^\alpha$.

Choose ρ by (4.3) for $m = \frac{N}{p}$. By adding multiples of N we can assume that $\rho \equiv 1 \left(\frac{N}{p}r\right)$.

$$\chi(\rho)\sum_{\nu(p)}\chi\left(\mu+\frac{Nr}{p}\nu\right) = \sum_{\nu(p)}\chi\left(\mu+\left(\rho\nu+\mu\frac{\rho-1}{\frac{Nr}{p}}\right)\frac{Nr}{p}\right) = \sum_{\nu(p)}\chi\left(\mu+\frac{Nr}{p}\nu\right),$$

and the statement follows. ■

Lemma (4.5)
Let χ be a primitive character of period N such that $2 \mid N$. Then for any $c \in \mathbb{Z}$ we have

$$\chi(c) = -\chi\left(c+\frac{N}{2}\right). \qquad \square$$

Proof
The statement is clear if $\chi(c) = 0$. Otherwise we may restrict our attention to $c = 1$ by multiplying with $\chi(n)$ for an appropriate odd number n and then by (4.3) on page 23 we have $\chi(1) \neq \chi\left(1+\frac{N}{2}\right)$. On the other hand $\chi\left(1+\frac{N}{2}\right)^2 = \chi(1) = 1$ and hence the statement follows. ■

We will need to know, which minimal period χ^2 has. The following lemma will give the answer.

Lemma (4.6)
Let χ be a primitive character with period N. Then the minimal period of χ^2 is

- N, if $\nu_2(N) = 0$,
- $\frac{N}{4}$, if $\nu_2(N) = 3$ and
- $\frac{N}{2}$, otherwise. □

Proof
For any prime p we have $\chi = \chi_p\widetilde{\chi}_p$ where the period of χ_p is $p^{\nu_p(N)}$ and the period of $\widetilde{\chi}_p$ is $Np^{-\nu_p(N)}$. Hence we may restrict our attention to powers of primes $N = p^\alpha$.

If $p \neq 2$ and $\alpha = 1$ we have $1 = \chi^2(1) \neq \chi^2(0) = 0$. Hence the period is not 1, thus it is p.

If $p \neq 2$ and $\alpha > 1$ there is $n \in \mathbb{Z}$ such that $\chi(n)$ is a primitive $p^{\alpha-1}$-th root of unity. Thus $\chi^2(n)$ is another one and χ^2 is primitive with period p^α.

There is no primitive Dirichlet character for $p = 2$ and $\alpha = 1$. Suppose $p = 2$ and $\alpha \in \{2, 3\}$. Then χ is real an thus χ^2 has minimal period 2.

There remains the case $p = 2$ and $\alpha > 3$. Then there is $n \in \mathbb{Z}$ such that $\chi(n)$ is primitive $2^{\alpha-2}$-th root of unity. Thus $\chi^2(n)$ is a primitive $2^{\alpha-3}$-th root of unity. Hence χ^2 is induced by a character with period $2^{\alpha-1}$ and this is the minimal period of χ^2. ■

—— *Character sums* ——

Here we define a character sum, which we use to calculate the second integral representation. All calculations are those of Kuß [5, p. 48]. Observe that we don't multiply by ν^{-2}.

Lemma (4.7)
Let χ be a primitive Dirichlet character with period N and suppose $\nu \mid N$. Suppose χ^2 is induced by a primitive character $\chi^{(2)}$ with period L and let LR by the least period of χ^2. Set $r := \frac{N}{(N,L\nu)}$. Given β, $\gamma \in \mathbb{Z}$ choose β^*, $\gamma^* \in \mathbb{Z}$ such that $\beta\beta^* \equiv \gamma\gamma^* \equiv 1\ (N\nu)$. If $m \in \mathbb{Z}$ we have

$$\begin{aligned} A_{\chi,\nu}(m) &:= \sum_{\substack{\beta (N\nu)^\times \\ \gamma (N\nu)^\times \\ \beta \equiv \gamma\ (\nu)}} \chi\left(\frac{\beta-\gamma}{\nu}\right) e^{2\pi\imath m \frac{\gamma^*-\beta^*}{N\nu}} \\ &= \begin{cases} \frac{\nu}{LR}\chi(-m)\chi^{(2)}(R)\overline{\chi^{(2)}}\left(\frac{LR\nu}{N}\right)\mu(r)\phi\left(\frac{R}{r}\right) G_{\overline{\chi}}^3 G_{\chi^{(2)}}, & \text{if } \frac{N}{LR} \mid \nu \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In particular $A_{\chi,\nu}(m) = \chi(m)A_{\chi,\nu}(1)$. □

Proof
We have

$$\begin{aligned} A_{\chi,\nu}(m) &:= \sum_{\substack{\beta (N\nu)^\times \\ \gamma (N\nu)^\times \\ \beta \equiv \gamma\ (\nu)}} \chi\left(\frac{\beta-\gamma}{\nu}\right) e^{2\pi\imath m \frac{\gamma^*-\beta^*}{N\nu}} \\ &= \sum_{\substack{\beta (N\nu)^\times \\ \gamma (N\nu)^\times \\ \beta \equiv \gamma\ (\nu)}} \chi(\beta\gamma)\chi\left(\frac{\gamma^*-\beta^*}{\nu}\right) e^{2\pi\imath m \frac{\gamma^*-\beta^*}{N\nu}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{\beta(N\nu)^\times \\ \gamma(N\nu)^\times \\ \beta\equiv\gamma\ (\nu)}} \overline{\chi}(\beta^*\gamma^*)\chi\left(\frac{\gamma^*-\beta^*}{\nu}\right) e^{2\pi\imath m\frac{\gamma^*-\beta^*}{N\nu}} \\
&= \sum_{\substack{\beta(N\nu) \\ \gamma(N\nu) \\ \beta\equiv\gamma\ (\nu)}} \overline{\chi}(\beta\gamma)\chi\left(\frac{\gamma-\beta}{\nu}\right) e^{2\pi\imath m\frac{\gamma-\beta}{N\nu}} \\
&= \sum_{\substack{\beta(N\nu) \\ r(N)}} \overline{\chi}(\beta(\beta+\nu r))\chi(r)e^{2\pi\imath m\frac{r}{N}} \\
&= \nu \sum_{\substack{\beta(N)^\times \\ r(N)^\times}} \overline{\chi}(\beta(\beta+\nu r))\chi(r)e^{2\pi\imath m\frac{r}{N}} \\
&= \nu \sum_{\substack{\beta(N)^\times \\ r(N)^\times}} \overline{\chi}(\beta r(\beta r+\nu r))\chi(r)e^{2\pi\imath m\frac{r}{N}} \\
&= \nu \sum_{\substack{\beta(N)^\times \\ r(N)^\times}} \overline{\chi}(\beta(\beta+\nu))\overline{\chi}(r)e^{2\pi\imath m\frac{r}{N}} \\
&= \nu\chi(m)G_{\overline{\chi}} \sum_{\beta(N)^\times} \overline{\chi}(\beta(\beta+\nu)),
\end{aligned}
$$

since χ is primitive. Furthermore

$$
G_\chi\overline{\chi}(\beta+\nu) = \sum_{\gamma(N)^\times} \chi(\gamma)e^{2\pi\imath(\beta+\nu)\frac{\gamma}{N}}.
$$

Hence we may continue

$$
\begin{aligned}
A_{\chi,\nu}(m) &= \nu\chi(m)G_{\overline{\chi}} \sum_{\beta(N)^\times} \overline{\chi}(\beta(\beta+\nu)) \\
&= \nu\chi(m)\frac{G_{\overline{\chi}}}{G_\chi} \sum_{\substack{\beta(N)^\times \\ \gamma(N)^\times}} \overline{\chi}(\beta)\chi(\gamma)e^{2\pi\imath(\beta+\nu)\frac{\gamma}{N}} \\
&= \nu\chi(-m)\frac{G_{\overline{\chi}}^2}{N} \sum_{\substack{\beta(N)^\times \\ \gamma(N)^\times}} \overline{\chi}(\beta)\chi(\gamma)e^{2\pi\imath(\beta+\nu)\frac{\gamma}{N}} \\
&= \nu\chi(-m)\frac{G_{\overline{\chi}}^2}{N} \sum_{\substack{\beta(N)^\times \\ \gamma(N)^\times}} \overline{\chi}(\beta\gamma^*)\chi(\gamma)e^{2\pi\imath(\beta\gamma^*+\nu)\frac{\gamma}{N}}
\end{aligned}
$$

$$
\begin{aligned}
&= \nu\chi(-m)\frac{G_{\overline{\chi}}^2}{N}\sum_{\substack{\beta(N)^\times\\ \gamma(N)^\times}} \overline{\chi}(\beta)\chi(\gamma)^2 e^{2\pi\imath\frac{\beta+\nu\gamma}{N}}\\
&= \nu\chi(-m)\frac{G_{\overline{\chi}}^3}{N}\sum_{\gamma(N)^\times} \chi(\gamma)^2 e^{2\pi\imath\frac{\nu\gamma}{N}}\\
&= \nu\chi(-m)\frac{G_{\overline{\chi}}^3}{N}\sum_{\substack{\gamma(LR)^\times\\ r(\frac{N}{LR})}} (\mathbb{1}_R\chi^{(2)})(\gamma) e^{2\pi\imath\frac{\nu(rLR+\gamma)}{N}}.
\end{aligned}
$$

The sum involving r vanishes if $\frac{N}{LR}\nmid\nu$. Otherwise by (4.2) on page 21 we have

$$
\begin{aligned}
A_{\chi,\nu}(m) &= \nu\chi(-m)\frac{G_{\overline{\chi}}^3}{LR}\sum_{\gamma(LR)^\times} (\mathbb{1}_R\chi^{(2)})(\gamma) e^{2\pi\imath\frac{\nu\gamma}{N}}\\
&= \nu\chi(-m)\frac{G_{\overline{\chi}}^3}{LR}\chi^{(2)}(R)\overline{\chi^{(2)}}\left(\frac{LR\nu}{N}\right)\mu(r)\phi\left(\frac{R}{r}\right)G_{\chi^{(2)}},
\end{aligned}
$$

where

$$
r = \frac{R}{(R, \frac{LR\nu}{N})} = \frac{N}{\frac{N}{R}(R, \frac{LR\nu}{N})} = \frac{N}{(N, L\nu)}. \qquad \blacksquare
$$

—— *Characters of* $\Gamma_{2,1}^{(p)*}(Np, \frac{N^2}{\kappa}p)$ ——

We already now that any Dirichlet character χ with period N induces a character on $\Gamma_{2,1}(N,N)$, hence on $\Gamma_{2,1}^{(p)}(Np, \frac{N^2}{\kappa}p)$ whenever p is a prime, N and p are coprime and $\kappa \mid N$ by means of $\chi(M) = \chi(\delta)$.

Lemma (4.8)
Let $\chi, \mathbb{N}, \kappa, p$ be as above. Then χ induces a character of $\Gamma_{2,1}^{(p)*}(Np, \frac{N^2}{\kappa}p)$ by means of

$$
\chi(M) = \begin{cases} \chi(\delta), & \text{if } M \in \Gamma_{2,1}^{(p)}\left(Np, \frac{N^2}{\kappa}p\right)\\ \sqrt{\overline{\chi}(p)}\,\chi(MH_p(N)^{-1}), & \text{otherwise,}\end{cases}
$$

where $\sqrt{\overline{\chi}(p)}$ is one fixed root $\square$

Proof
Since $\Gamma_{2,1}^{(p)*}(Np, \frac{N^2}{\kappa}p)$ is a normal extension of $\Gamma_{2,1}^{(p)}(Np, \frac{N^2}{\kappa}p)$ and

$$
(H_p(N)MH_p(N)^{-1})_{4,4} \equiv M_{4,4}\ (N),
$$

if $M \in \Gamma_{2,1}^{(p)}(Np, \frac{N^2}{\kappa}p)$, we only have to check whether

$$\chi(H_p(N)^2) = \chi(H_p(N))^2.$$

This is true since

$$\begin{aligned}(H_p(N)^2)_{4,4} &= (-1 + p^*py)(1 - p^*p) + p(x - p^*y)^2 \\ &\equiv p^* \quad (N).\end{aligned}$$

■

We introduce a notation to distinguish different roots.

Definition (4.9)

Let $\chi, \mathbb{N}$, κ, p be as above. Suppose $p \equiv 1$ (N). Let χ^+ be the character of (4.8) where $\chi^+(H_p(N)) = 1$ and given k let χ^{k-} be the character such that $\chi^{k-}(H_p(N)) = (-1)^k$. If k is clear by the context we write $\chi^- = \chi^{k-}$. □

§ 5 Siegel modular forms

In this section we introduce twists of Siegel modular forms in $[\Gamma_{2,1}^{(p)*}(p,p),k,\mathbb{1}_1^{k-}]_0$. The special case $p=1$ was treated by Kohnen, Krieg and Sengupta in [9].

We use the following conventions which are valid if we don't state differently. Let p be a prime, χ be a character of period N, where $p\equiv 1\ (N)$. Moreover let $f\in[\Gamma_{2,1}^{(p)*}(p,p),k,\mathbb{1}_1^{k-}]_0$ be a Siegel cusp form with Fourier-Jacobi series $f(Z)=\sum_{m=1}^{\infty} f_m(Z_1,w)e^{2\pi\imath mz}$.

First we state a basic fact.

Lemma (5.1)
Let f be a meromorphic 1-periodic function on $\mathbb{H}_n$ which admits a Fourier-Jacobi series $f(Z)=\sum_{m=1}^{\infty} f_m(Z_1,w)e^{2\pi\imath mz}$. Let $t\in\mathbb{N}$. Then $f\Big|M_{\frac{1}{t}}=f$ if and only if $f_m\equiv 0$ whenever $t\nmid m$. □

Proof
We have

$$f(Z)=\sum_{m\in\mathbb{Z}} f_m(Z_1,w)e^{2\pi\imath mz} \qquad \text{and}$$
$$f\Big|M_{\frac{1}{t}}(Z)=\sum_{m\in\mathbb{Z}} f_m(Z_1,w)e^{2\pi\imath mz}e^{2\pi\imath\frac{m}{t}}.$$

Due to the uniqueness of the Fourier-Jacobi series we have equality if and only if $e^{2\pi\imath\frac{m}{t}}=1$ hence $t\mid m$ whenever f_m doesn't vanish. ■

Now we are ready to make the following

Definition (5.2)
Let $f\in[\Gamma_{2,1}^{(p)*}(p,p),k,\mathbb{1}_1^{k-}]_0$ be a Siegel cusp form with Fourier-Jacobi series $f(Z)=\sum_{m=1}^{\infty} f_m(Z_1,w)e^{2\pi\imath mz}$. Then the twist of f by χ is defined to be

$$f_\chi:=\sum_{m=1}^{\infty}\chi(m)f_m. \qquad \square$$

Proposition (5.3)
We have

$$f_\chi=\frac{1}{N}\sum_{\nu,\,\mu(N)}\chi(\nu)e^{-2\pi\imath\frac{\nu\mu}{N}}f\Big|M_{\frac{\mu}{N}}.$$

f_χ is a Siegel cusp form in $[\Gamma_{2,1}^{(p)*}(Np,N^2p),\,k,\,(\chi^2)^{k-}]_0$. □

Proof
To proof the first representation we calculate

$$\begin{aligned}
f_\chi(Z) &= \sum_{m=1}^{\infty} \chi(m) f_m(Z_1, w) e^{2\pi imz} \\
&= \sum_{\nu(N)} \sum_{\substack{m\in\mathbb{N} \\ m\equiv\nu\ (N)}} \chi(\nu) f_m(Z_1, w) e^{2\pi imz} \\
&= \frac{1}{N} \sum_{\nu(N)} \chi(\nu) f(Z_1, w) e^{2\pi imz} \sum_{\mu(N)} e^{2\pi\imath(m-\nu)\frac{\mu}{N}} \\
&= \frac{1}{N} \sum_{\nu,\,\mu(N)} \chi(\nu) e^{-2\pi\imath\frac{\nu\mu}{N}} f \Big| M_{\frac{\mu}{N}}\ (Z).
\end{aligned}$$

For the second statement consider $M \in \Gamma_{n,1}^{(p)}(Np, N^2p)$. Using the notation of section 3 we see

$$\begin{aligned}
\tilde{M} = M_{\frac{\mu}{N}} M M_{\frac{-\mu\delta^2}{N}} &\in \Gamma_{2,1}^{(p)}(Np, N^2p) \\
&\subseteq \Gamma_{2,1}^{(p)*}(p, p).
\end{aligned}$$

The matrix $\tilde{M}$ has lower right entry $\delta - \frac{\mu\delta^2}{N}\gamma \equiv \delta\ (N)$. Considering $\alpha\delta \equiv 1\ (N)$ we have

$$\begin{aligned}
f_\chi|M &= \frac{1}{N} \sum_{\nu,\,\mu(N)} \chi(\nu) e^{-2\pi\imath\frac{\nu\mu}{N}} f \Big| M_{\frac{\mu}{N}} M \\
&= \frac{1}{N} \sum_{\nu,\,\mu(N)} \chi(\nu) e^{-2\pi\imath\frac{\nu\mu}{N}} f \Big| M_{\frac{\mu\delta^2}{N}} \\
&= \frac{1}{N} \chi(\delta^2) \sum_{\nu,\,\mu(N)} \chi(\nu\alpha^2) e^{-2\pi\imath\frac{\nu\mu}{N}\alpha^2\delta^2} f \Big| M_{\frac{\mu\delta^2}{N}} \\
&= \chi(M)^2 f_\chi,
\end{aligned}$$

since if ν runs through a set of representatives of $\mathbb{Z}/N\mathbb{Z}$ so does $\alpha^2\nu$ and analogously for $\delta^2\mu$.

There remains to treat $H_p(N)$. The matrix $M_{\frac{\mu}{N}} H_p(N) M_{\frac{-\mu(x-p^*y)}{N}}$ has upper right submatrix L where

$$L = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & \frac{\mu p(x-p^*y)}{N}(1-p^*p) \\ \frac{\mu}{N} p(1-p^*p) & \frac{\mu}{N} p(x-p^*y)(1-p^*p) \end{pmatrix}.$$

Recall that $(x - p^*y) \equiv p^* \equiv 1\ (N)$. The matrix $\sqrt{p}^{-1}L$ has integral entries. Thus the matrix $H_p(N)^{-1} M_{\frac{\mu}{N}} H_p(N) M_{\frac{-\mu(x-p^*y)}{N}}$ is in $\Gamma_{2,1}^{(p)}(Np, N^2p)$. We obtain

$$\begin{aligned} f_\chi \,|H_p(N) &= \frac{1}{N} \sum_{\nu,\,\mu(N)} \chi(\nu) e^{-2\pi\imath \frac{\nu\mu}{N}} f \left| M_{\frac{\mu}{N}} H_p(N) \right. \\ &= (-1)^k \frac{1}{N} \sum_{\nu,\,\mu(N)} \chi(\nu) e^{-2\pi\imath \frac{\nu\mu}{N}} f \left| H_p(N)^{-1} M_{\frac{\mu}{N}} H_p(N) M_{\frac{-\mu(x-p^*y)}{N}} M_{\frac{\mu(x-p^*y)}{N}} \right. \\ &= (-1)^k f_\chi. \end{aligned}$$

Hence we have

$$f_\chi \in [\Gamma_{2,1}^{(p)*}(Np, N^2p),\, k,\, (\chi^2)^{k-}].$$

Finally f_χ is a cusp form since in the Fourier series of $f \left| M_{\frac{\mu}{N}} \right.$ there are only non vanishing coefficients for positive definite matrices. ■

We can refine the representation if χ is primitive.

Lemma (5.4)
Let χ be primitive. Then we have

$$f_\chi = \frac{1}{G_{\overline{\chi}}} \sum_{\mu(N)} \overline{\chi}(\mu) f \left| M_{\frac{\mu}{N}} \right. \qquad \square$$

Proof
Using $G_\chi G_{\overline{\chi}} = \chi(-1)N$ we obtain using (5.3)

$$\begin{aligned} f_\chi &= \frac{1}{N} \sum_{\nu,\,\mu(N)} \chi(\nu) e^{-2\pi\imath \frac{\nu\mu}{N}} f \left| M_{\frac{\mu}{N}} \right. \\ &= \frac{1}{G_{\overline{\chi}}} \sum_{\mu(N)} \overline{\chi}(\mu) f \left| M_{\frac{\mu}{N}} \right. . \end{aligned} \qquad ■$$

This representation is useful to prove

Proposition (5.5)
For $\nu \in \mathbb{N}$ we have $f_\chi \,|W_{N\nu p} \in [\Gamma_{2,1}^{(p)*}(N\nu p, (N\nu)^2 p),\, k,\, 0,\, \left(\overline{\chi}^2\right)^{k-}]_0$. Furthermore we have

$$f_\chi \,|W_{Np} = \frac{G_\chi^2}{N} f_{\overline{\chi}}. \qquad \square$$

Proof
To prove the first statement consider $M \in \Gamma_{2,1}^{(p)}(N\nu p, (N\nu)^2 p)$. Then we see that $\tilde{M} = W_{N\nu p} M W_{N\nu p}^{-1} \in \Gamma_{2,1}^{(p)}(N\nu p, (N\nu)^2 p)$ and the lower right entry of $\tilde{M}$ is α, thus $\chi(\tilde{M}) = \overline{\chi}(M)$. Furthermore the lower right entry of $H_p(N\nu)^{-1} W_{N\nu p} H_p(N\nu) W_{N\nu p}^{-1}$ is $(1 - p^* p)^2/(N\nu) + p^{*2} p^3 \equiv 1\ (N)$.

For the second statement consider μ such that $(\mu, N) = 1$ and define μ^* by means of $\mu^* \mu p^2 \equiv 1\ (N)$. Then

$$M_{\frac{\mu}{N}} W_{Np} M_{\frac{\mu^*}{N}} = \left(\begin{array}{cc|cc} & & 1 & \\ & -\mu p & & \frac{1-\mu^*\mu p^2}{Np} \\ \hline -1 & & & \\ & -Np & & -\mu^* p \end{array}\right) \in \Gamma_{2,1}^{(p)}(Np, Np).$$

Hence

$$\begin{aligned} f\,|W_{Np} &= \frac{1}{G_{\overline{\chi}}} \sum_{\mu(N)^\times} \overline{\chi}(\mu) f \left| M_{\frac{\mu}{N}} W_{Np} \right. \\ &= \frac{1}{G_{\overline{\chi}}} \sum_{\mu(N)^\times} \overline{\chi}(\mu) f \left| M_{\frac{-\mu^*}{N}} \right. \\ &= \frac{\chi(-1)}{G_{\overline{\chi}}} \sum_{\mu(N)^\times} \chi(-\mu^*) f \left| M_{\frac{-\mu^*}{N}} \right. \\ &= \frac{\chi(-1)}{G_{\overline{\chi}}} G_\chi f_{\overline{\chi}} = \frac{G_\chi^2}{N} f_{\overline{\chi}}. \end{aligned}$$ ■

We want to prove one further property.

Proposition (5.6)
Let $q \mid N$ be a prime. Then we have

$$\sum_{\nu(q)} f_\chi \left| M^{\mathrm{tr}}_{(Np)^2 \frac{\nu}{q}} \right. = 0.$$ □

Proof

We have $W_{Np} M^{\text{tr}}_{(Np)^2 \frac{\nu}{q}} W_{Np}^{-1} = M_{-\frac{\nu}{q}}$. Hence

$$
\begin{aligned}
& \sum_{\nu(q)} f_\chi \left| M^{\text{tr}}_{(Np)^2 \frac{\nu}{q}} \right. \\
&= \frac{G_\chi^2}{N} \sum_{\nu(q)} f_{\overline{\chi}} \left| W_{Np} M^{\text{tr}}_{(Np)^2 \frac{\nu}{q}} \right. \\
&= \frac{G_\chi}{N} \sum_{\nu(q)} \sum_{\mu(N)} \chi(\mu) f \left| M_{\frac{\mu}{N}} M_{-\frac{\nu}{q}} W_{Np} \right. \\
&= \frac{G_\chi}{N} \sum_{\nu(q)} \sum_{\mu(N)} \chi(\mu + N\frac{\nu}{q}) f \left| M_{\frac{\mu}{N}} W_{Np} \right. \\
&= 0,
\end{aligned}
$$

if we sum over ν and use (4.4) on page 23. ■

§ 6 Eisensteinseries for $\Gamma_{n,1}\left(Nt, \frac{(Nt)^2}{\kappa}\right)$

This section follows the lines of [5], where the special case $p = 1$ was treated and Gritsenko's results for the special case $N = 1$ in [2].

Let χ be a Dirichlet character with period N, let p be a prime, such that $p \equiv 1\ (N)$ and suppose $\kappa \mid N$.

Definition (6.1)
Given $Z \in \mathbb{H}_2$, $s \in \mathbb{C}$ where $\mathfrak{Re}(s) > 2$ the Klingen-Eisensteinseries for $\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right)$ is given by

$$E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s) := \sum_{M:\Gamma_{2,1}^{(p)}\backslash\Gamma_{2,1}^{(p)*}\left(Np,\frac{N^2}{\kappa}p\right)} \chi^+(M)\left(\frac{\det\mathfrak{Im}(M\langle Z\rangle)}{\det\mathfrak{Im}(M\langle Z\rangle)_1}\right)^s.$$

Its completion is given by

$$\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s) = \pi^{-s}p^{\frac{3}{2}s}(1+p^{-s})N^{2s}\Gamma(s)L(2s,\chi)E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s). \qquad \square$$

We first prove the

Proposition (6.2)
The series $E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)$ is well defined and converges locally absolutely uniformly for $Z \in \mathbb{H}_2$, $s \in \mathbb{C}$ where $\mathfrak{Re}(s) > 2$. For fixed s we have

$$E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(\cdot,s) \in [\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right), 0, \overline{\chi}^+]. \qquad \square$$

Proof
We use (1.3) on page 6. The expression

$$\frac{\det\mathfrak{Im}(M\langle Z\rangle)_1}{\det\mathfrak{Im}(M\langle Z\rangle)} = P_Z[\lambda],$$

where λ is the last row of M is indepentend of the choice of representatives, hence the series is well defined.

A system of representatives of $\Gamma_{2,1}^{(p)}\backslash\Gamma_{2,1}^{(p)*}\left(Np, \frac{N^2}{\kappa}p\right)$ is given by (3.9) on page 17 and hence we get

$$
\begin{aligned}
&E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)\\
&= \sum_{\substack{\lambda\in\mathbb{Z}^{4\times}\\ (\lambda_2,p)=1\\ (\lambda_4,N)=1}} p^{-2s}\chi(\lambda_4)P_Z[\begin{pmatrix} N\lambda_1\\ \frac{N^2}{\kappa}\lambda_2\\ N\lambda_3\\ \lambda_4\end{pmatrix}]^{-s} + \sum_{\substack{\lambda\in\mathbb{Z}^{4\times}\\ (\lambda_4,Np)=1}} \chi(\lambda_4)P_Z[\begin{pmatrix} Np\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ Np\lambda_3\\ \lambda_4\end{pmatrix}]^{-s}\\
&\quad + \sum_{\substack{\lambda\in\mathbb{Z}^{4\times}\\ (\lambda_1,p)=1\\ (\lambda_4,N)=1}} p^{-s}\chi(\lambda_4)P_Z[\begin{pmatrix} N\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ N\lambda_3\\ \lambda_4\end{pmatrix}]^{-s} + \sum_{\substack{\lambda\in\mathbb{Z}^{4\times}\\ (\lambda_3,p)=1\\ (\lambda_4,N)=1}} p^{-s}\chi(\lambda_4)P_Z[\begin{pmatrix} Np\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ N\lambda_3\\ \lambda_4\end{pmatrix}]^{-s}.
\end{aligned}
$$

The Epstein ζ-function given in (1.1) on page 5 gives rise to the dominating series

$$4\zeta(\mathfrak{Re}(s),0,0,P_Z) = \sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}} 4P_Z[\lambda]^{-\mathfrak{Re}(s)},$$

which converges locally absolutely uniformly if $\mathfrak{Re}(s) > 2$.

Due to the absolute convergence of $E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)$ for $M' \in \Gamma^{(p)*}_{2,1}\left(Np, \frac{N^2}{\kappa}p\right)$ we have

$$
\begin{aligned}
E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)\,|M' &= \sum_{M:\Gamma^{(p)}_{2,1}\backslash\Gamma^{(p)*}_{2,1}\left(Np,\frac{N^2}{\kappa}p\right)} \chi^+(M)\left(\frac{\det \mathfrak{Im}(M\langle M'\langle Z\rangle\rangle)}{\det \mathfrak{Im}(M\langle M'\langle Z\rangle\rangle)_1}\right)^s\\
&= \sum_{M:\Gamma^{(p)}_{2,1}\backslash\Gamma^{(p)*}_{2,1}\left(Np,\frac{N^2}{\kappa}p\right)} \overline{\chi}^+(M')\chi^+(MM')\left(\frac{\det \mathfrak{Im}(MM'\langle Z\rangle)}{\det \mathfrak{Im}(MM'\langle Z\rangle)_1}\right)^s\\
&= \overline{\chi}^+(M')E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s).
\end{aligned}
$$

■

Later we need another modular form deduced from the Eisenstein series.

Proposition (6.3)

Let $\nu \in \mathbb{N}$. For fixed s we have

$$E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(\cdot,s)\,|W_{N\nu p} \in [\Gamma^{(p)*}_{2,1}\left(N\nu p,(N\nu)^2p\right),0,\chi^+].$$

□

Proof

This is analogous to the proof of the first part of (5.5) on page 31. ■

— *Meromorphic continuation and growth* —

The completed Eisenstein series has a meromorphic continuation. This is the statement of

Proposition (6.4)
The completed Eisenstein series satisfies

$$\begin{aligned}
&\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)\\
&= \pi^{-s}p^{\frac{3}{2}s}N^{2s}\Gamma(s)\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(\lambda_4)\left(p^{-s}P_Z[\begin{pmatrix} N\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ N\lambda_3\\ \lambda_4\end{pmatrix}]^{-s} + P_Z[\begin{pmatrix} Np\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ Np\lambda_3\\ \lambda_4\end{pmatrix}]^{-s}\right)\\
&= N^{-2s}\sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}}\chi(\delta)\left(p^{\frac{1}{2}s-1}\sum_{h(p)}\zeta^*\left(s,\left(0,\frac{\kappa h}{p},0,0\right)^{\mathrm{tr}},\left(\frac{\alpha}{N},\frac{p\beta}{\kappa},\frac{\gamma}{N},\frac{\delta}{N^2}\right)^{\mathrm{tr}},P_Z\right)\right.\\
&\qquad\left. + p^{-\frac{1}{2}s}\sum_{g(p)}\zeta^*\left(s,0,\left(\frac{\alpha}{N},\frac{\beta}{\kappa},\frac{\gamma}{N},\frac{p\delta+N^2g}{N^2p}\right)^{\mathrm{tr}},P_Z\right)\right)\\
&= N^{-2s}\sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}}\chi(\delta)\\
&\quad\left(p^{-\frac{3}{2}s}\sum_{h_1,h_2,h_3(p)}\zeta^*\left(s,0,\left(\frac{p\alpha+Nh_1}{Np},\frac{\beta}{\kappa},\frac{p\gamma+Nh_2}{Np},\frac{p\delta+N^2h_3}{N^2p}\right)^{\mathrm{tr}},P_Z\right)\right.\\
&\quad\left. + p^{-\frac{1}{2}s}\sum_{g(p)}\zeta^*\left(s,0,\left(\frac{\alpha}{N},\frac{\beta}{\kappa},\frac{\gamma}{N},\frac{p\delta+N^2g}{N^2p}\right)^{\mathrm{tr}},P_Z\right)\right).
\end{aligned}$$

Moreover the completed Eisenstein series $\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(\cdot,s)$ has a meromorphic continuation to the whole s-plane. If χ is not the trivial character, this continuation is holomorphic. Otherwise it is holomorphic except a simple pole at $s=2$ with residuum $\frac{2\kappa\phi(N)}{N}$. □

Proof
Using (6.2) we get

$$(1+p^{-s})L(2s,\chi)E^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)$$

$$
\begin{aligned}
&= \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_2,p)=1\\ (\lambda_4,N)=1}} p^{-s}\chi(\lambda_4 m)P_Z[\begin{pmatrix} Np\lambda_1 m\\ \frac{N^2}{\kappa}p\lambda_2 m\\ Np\lambda_3 m\\ p\lambda_4 m\end{pmatrix}]^{-s} + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_4,Np)=1}} p^{-s}\chi(\lambda_4 m)P_Z[\begin{pmatrix} Np\lambda_1 m\\ \frac{N^2}{\kappa}p\lambda_2 m\\ Np\lambda_3 m\\ \lambda_4 m\end{pmatrix}]^{-s}\\
&\quad + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_1,p)=1\\ (\lambda_4,N)=1}} p^{-s}\chi(\lambda_4 m)P_Z[\begin{pmatrix} N\lambda_1 m\\ \frac{N^2}{\kappa}p\lambda_2 m\\ N\lambda_3 m\\ \lambda_4 m\end{pmatrix}]^{-s} + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_3,p)=1\\ (\lambda_4,N)=1}} p^{-s}\chi(\lambda_4 m)P_Z[\begin{pmatrix} Np\lambda_1 m\\ \frac{N^2}{\kappa}p\lambda_2 m\\ N\lambda_3 m\\ \lambda_4 m\end{pmatrix}]^{-s}\\
&\quad + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_2,p)=1\\ (\lambda_4,N)=1}} \chi(\lambda_4 m)P_Z[\begin{pmatrix} Np\lambda_1 m\\ \frac{N^2}{\kappa}p\lambda_2 m\\ Np\lambda_3 m\\ p\lambda_4 m\end{pmatrix}]^{-s} + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_4,Np)=1}} \chi(\lambda_4 m)P_Z[\begin{pmatrix} Np\lambda_1 m\\ \frac{N^2}{\kappa}p\lambda_2 m\\ Np\lambda_3 m\\ \lambda_4 m\end{pmatrix}]^{-s}\\
&\quad + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_1,p)=1\\ (\lambda_4,N)=1}} \chi(\lambda_4 m)P_Z[\begin{pmatrix} Np\lambda_1 m\\ \frac{N^2}{\kappa}p^2\lambda_2 m\\ Np\lambda_3 m\\ p\lambda_4 m\end{pmatrix}]^{-s} + \sum_{\substack{m\geq 1\\ \lambda\in\mathbb{Z}^{4\times}\\ (\lambda_3,p)=1\\ (\lambda_4,N)=1}} \chi(\lambda_4 m)P_Z[\begin{pmatrix} Np^2\lambda_1 m\\ \frac{N^2}{\kappa}p^2\lambda_2 m\\ Np\lambda_3 m\\ p\lambda_4 m\end{pmatrix}]^{-s}\\
&= p^{-s}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}} \chi(\lambda_4)P_Z[\begin{pmatrix} N\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ N\lambda_3\\ \lambda_4\end{pmatrix}]^{-s} + \sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}} \chi(\lambda_4)P_Z[\begin{pmatrix} Np\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ Np\lambda_3\\ \lambda_4\end{pmatrix}]^{-s},
\end{aligned}
$$

and this yields the first statement. Using this we may continue

$$
\begin{aligned}
&\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,s)\\
&= \pi^{-s}p^{\frac{3}{2}s}N^{2s}\Gamma(s)\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(\lambda_4)\left(p^{-s}P_Z[\begin{pmatrix} N\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ N\lambda_3\\ \lambda_4\end{pmatrix}]^{-s} + P_Z[\begin{pmatrix} Np\lambda_1\\ \frac{N^2}{\kappa}p\lambda_2\\ Np\lambda_3\\ \lambda_4\end{pmatrix}]^{-s}\right)\\
&= \pi^{-s}p^{\frac{3}{2}s}N^{-2s}\Gamma(s)\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(\lambda_4)\left(p^{-s}P_Z[\begin{pmatrix} \frac{1}{N}\lambda_1\\ \frac{p}{\kappa}\lambda_2\\ \frac{1}{N}\lambda_3\\ \frac{1}{N^2}\lambda_4\end{pmatrix}]^{-s} + p^{-2s}P_Z[\begin{pmatrix} \frac{1}{N}\lambda_1\\ \frac{1}{\kappa}\lambda_2\\ \frac{1}{N}\lambda_3\\ \frac{1}{N^2p}\lambda_4\end{pmatrix}]^{-s}\right)
\end{aligned}
$$

$$= \pi^{-s} p^{\frac{1}{2}s} N^{-2s} \Gamma(s)$$

$$\sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}} \chi(\delta) \left(p^{-1} \sum_{h(p)} \zeta\left(s, \left(0, \frac{\kappa h}{p}, 0, 0\right)^{\mathrm{tr}}, \left(\frac{\alpha}{N}, \frac{p\beta}{\kappa}, \frac{\gamma}{N}, \frac{\delta}{N^2}\right)^{\mathrm{tr}}, P_Z\right)\right.$$

$$\left. + p^{-s} \sum_{g(p)} \zeta\left(s, 0, \left(\frac{\alpha}{N}, \frac{\beta}{\kappa}, \frac{\gamma}{N}, \frac{p\delta + N^2 g}{N^2 p}\right)^{\mathrm{tr}}, P_Z\right)\right)$$

$$= N^{-2s} \sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}} \chi(\delta) \left(p^{\frac{1}{2}s-1} \sum_{h(p)} \zeta^*\left(s, \left(0, \frac{\kappa h}{p}, 0, 0\right)^{\mathrm{tr}}, \left(\frac{\alpha}{N}, \frac{p\beta}{\kappa}, \frac{\gamma}{N}, \frac{\delta}{N^2}\right)^{\mathrm{tr}}, P_Z\right)\right.$$

$$\left. + p^{-\frac{1}{2}s} \sum_{g(p)} \zeta^*\left(s, 0, \left(\frac{\alpha}{N}, \frac{\beta}{\kappa}, \frac{\gamma}{N}, \frac{p\delta + N^2 g}{N^2 p}\right)^{\mathrm{tr}}, P_Z\right)\right)$$

Analogously we may deduce the third representation.

Now we can apply (1.2) on page 6 and deduce, that we always have a meromorphic continuation to the whole s-plane. At most it has a simple pole at $s = 2$ with residuum

$$N^{-4} \sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}} \chi(\delta)(1+1) = \frac{2\kappa}{N} \sum_{\delta(N)} \chi(\delta) = \begin{cases} \frac{2\kappa\phi(N)}{N} & \text{, if } \chi \text{ is trivial} \\ 0 & \text{, otherwise.} \end{cases}$$

Thus the pole at $s = 2$ vanishes if χ is not trivial. ■

Our next step is to establish some bound on the growth of $\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(\cdot, s)$. To do so we need the following

Lemma (6.5)
Let $\theta_0 \in \mathbb{R}_+$ and define

$$C_{\theta_0} := 1 + 2 \sum_{\lambda \geq 1} e^{-\lambda^2 \theta_0}.$$

For $m \in \mathbb{N}$, $u \in \mathbb{R}^m$ there exists a non-negative constant $C(u)$ depending on u, such that

$$\sum_{\substack{\lambda \in \mathbb{Z}^m\\ \lambda + u \neq 0}} e^{-\|\lambda + u\|_2^2 \theta} \leq C_{\theta_0}^m \cdot e^{-C(u)\theta} \qquad \text{for all } \theta \geq \theta_0$$

$$\sum_{\lambda \in \mathbb{Z}^m} e^{-\|\lambda + u\|_2^2 \theta} \leq (C_{\theta_0} - 1)^m,$$

where $\|\cdot\|_2$ is the euclidean norm. □

Proof
Let $u = (u_1, \ldots, u_m)^{\mathrm{tr}}$. Then the second estimate is an immediate consequence of

$$\sum_{\lambda\in\mathbb{Z}^m} e^{-\|\lambda+u\|_2^2\theta} = \prod_{1\le\nu\le m}\left(\sum_{\lambda\in\mathbb{Z}} e^{-(\lambda+u_\nu)^2\theta}\right) \le \prod_{1\le\nu\le m}\left(2\sum_{\lambda\ge 0} e^{-\lambda^2\theta}\right) = (C_{\theta_0}-1)^m.$$

We will prove the first estimate by induction on m. Let $m = 1$. Then if $u \in \mathbb{Z}$ we get

$$\begin{aligned}\sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda+u\neq 0}} e^{-\|\lambda+u\|_2^2\theta} &= \sum_{\lambda\in\mathbb{Z}\setminus\{0\}} e^{-\lambda^2\theta} = 2\sum_{\lambda\ge 1} e^{-\lambda^2\theta} = 2\sum_{\lambda\ge 0} e^{-(\lambda+1)^2\theta}\\ &= 2\sum_{\lambda\ge 0} e^{-(\lambda^2+2\lambda+1)\theta} \le 2e^{-\theta}\sum_{\lambda\ge 0} e^{-\lambda^2\theta} < C_{\theta_0}e^{-\theta}.\end{aligned}$$

Now consider $u \notin \mathbb{Z}$ and choose $\mu \in \mathbb{Z}$ such that $|u-\mu|$ is minimal. We have $|u-\mu| > 0$. Then

$$\begin{aligned}\sum_{\substack{\lambda\in\mathbb{Z}\\ \lambda+u\neq 0}} e^{-\|\lambda+u\|_2^2\theta} &= \sum_{\lambda\in\mathbb{Z}} e^{-(\lambda+|u-\mu|)^2\theta} \le e^{-|u-\mu|^2\theta} + 2\sum_{\lambda\ge 1} e^{-(\lambda+|u-\mu|)^2\theta}\\ &\le e^{-|u-\mu|^2\theta}\cdot\left(1+2\sum_{\lambda\ge 1} e^{-\lambda^2\theta}\right) < C_{\theta_0}e^{-|u-\mu|^2\theta}.\end{aligned}$$

Now assume $m > 1$ and write $\lambda = \left(\begin{smallmatrix}\lambda_r\\ \lambda_m\end{smallmatrix}\right)$, $u = \left(\begin{smallmatrix}u_r\\ u_m\end{smallmatrix}\right)$ where $\lambda_r \in \mathbb{Z}^{m-1}$, $u_r \in \mathbb{R}^{m-1}$, $\lambda_m \in \mathbb{Z}$ and $u_m \in \mathbb{R}$. Then by induction we have

$$\begin{aligned}\sum_{\substack{\lambda\in\mathbb{Z}^m\\ \lambda+u\neq 0}} e^{-\|\lambda+u\|_2^2\theta} &= \sum_{\substack{\lambda\in\mathbb{Z}^m\\ \lambda_r+u_r\neq 0}} e^{-\|\lambda+u\|_2^2\theta} + \sum_{\substack{\lambda\in\mathbb{Z}^m\\ \lambda_r+u_r=0\\ \lambda_m+u_m\neq 0}} e^{-\|\lambda+u\|_2^2\theta}\\ &= \sum_{\substack{\lambda_r\in\mathbb{Z}^{m-1}\\ \lambda_r+u_r\neq 0}} e^{-\|\lambda_r+u_r\|_2^2\theta}\cdot\sum_{\lambda_m\in\mathbb{Z}} e^{-\|\lambda_m+u_m\|_2^2\theta} + \sum_{\substack{\lambda_m\in\mathbb{Z}\\ \lambda_m+u_m\neq 0}} e^{-\|\lambda_m+u_m\|_2^2\theta}\\ &\le C_{\theta_0}^{m-1}e^{-C(u_r)\theta}(C_{\theta_0}-1) + C_{\theta_0}^{m-1}e^{-C(u_m)\theta}\\ &\le C_{\theta_0}^{m}e^{-\min(C(u_r),C(u_m))\theta}\end{aligned}$$

■

Using this lemma we now can prove

Proposition (6.6)
Let $\delta = 1$ if χ is the trivial character, otherwise $\delta = 0$. Then for $M \in \Gamma_2$ and $Z \in \mathcal{F}_2$ and s in a compact set

$$\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(M\langle Z\rangle, s) - \delta\frac{2\kappa\phi(N)}{N(s-n)}$$

is bounded by a power of $\mathrm{tr}(Y)$. □

Proof
We use the integral presentation of the Epstein ζ-function given by equation (1.4) on page 7. Inserting this into the last representation given in (6.4) on page 36, we obtain

$$\begin{aligned}
&\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z, s) - \delta\frac{2\kappa\phi(N)}{N(s-n)} \\
&= N^{-2s}\sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}}\chi(\delta) \\
&\qquad\left(p^{-\frac{3}{2}s}\sum_{h_1,h_2,h_3(p)} I\left(\left(\frac{p\alpha+Nh_1}{Np}, \frac{\beta}{\kappa}, \frac{p\gamma+Nh_2}{Np}, \frac{p\delta+N^2h_3}{N^2p}\right)^{\mathrm{tr}}\right)\right. \\
&\qquad\left.+p^{-\frac{1}{2}s}\sum_{g(p)} I\left(\left(\frac{\alpha}{Np}, \frac{\beta}{\kappa}, \frac{\gamma}{Np}, \frac{p\delta+N^2g}{N^2p}\right)^{\mathrm{tr}}\right)\right) \\
&\quad+\left(N^{4-2s}\left(p^{-\frac{3}{2}s+3}+p^{-\frac{1}{2}s+1}\right)-2\right)\delta\frac{\kappa\phi(N)}{N(s-n)},
\end{aligned}$$

where

$$\begin{aligned}
I(v) = &\int_1^\infty \sum_{\substack{\lambda\in\mathbb{Z}^4\\ \lambda+v\neq 0}} e^{-\pi\theta P_{M\langle Z\rangle}[\lambda+v]}\theta^{s-1}d\theta \\
&+\int_1^\infty \sum_{\lambda\in\mathbb{Z}\setminus\{0\}} e^{-\pi\theta P_{M\langle Z\rangle}[\lambda]+2\pi\imath v^{\mathrm{tr}}\lambda}\theta^{n-s-1}d\theta.
\end{aligned}$$

The expression $\left(N^{4-2s}\left(p^{-\frac{3}{2}s+3}+p^{-\frac{1}{2}s+1}\right)-2\right)\delta\frac{\kappa\phi(N)}{N(s-n)}$ yields an entire function. Because $n-s$ varies in a compact set, if s does, it suffices to prove that

$$\int_1^\infty \sum_{\substack{\lambda\in\mathbb{Z}^{2n}\\ \lambda+v\neq 0}} e^{-\pi\theta P_{M\langle Z\rangle}[\lambda+v]}\theta^{s-1}d\theta \tag{5}$$

is bounded by a power of $\mathrm{tr}(Y)^{c(u)}$ for all $Z \in \mathcal{F}_n$.

Since $Z \in \mathcal{F}_n$ by [1] we have bounds for the minimal and maximal eigenvalue $e_{\min}$ and $e_{\max}$ of Y.

$$e_{\min} \geq e_1 \quad \text{and} \quad e_{\max} \leq e_2 \mathrm{tr}(Y)$$

with appropriate constants e_1, e_2 which depend only on n. Let $a := \min(e_{\min}, e_{\max}^{-1})$.

Let $\lambda_1, \lambda_2 \in \mathbb{Z}^n$, $u_1, u_2 \in \mathbb{R}^n$, $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then we get

$$\begin{aligned} P_Z[\lambda+u] &= \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \left[\begin{pmatrix} I_n & 0 \\ X & I_n \end{pmatrix} \begin{pmatrix} \lambda_1 + u_1 \\ \lambda_2 + u_2 \end{pmatrix} \right] \\ &= Y[\lambda_1 + u_1] + Y^{-1}[X(\lambda_1 + u_1) + \lambda_2 + u_2]. \end{aligned}$$

If we consider $\theta \geq 1$, the series of expression (5) with $\lambda + u$ is running through all $M^{\mathrm{tr}}(\lambda + v)$ can now be estimated by (6.5) on page 38 with $\theta_0 = \pi a \theta$.

$$\begin{aligned} \sum_{\substack{\lambda \in \mathbb{Z}^{2n} \\ \lambda + u \neq 0}} e^{-\pi\theta P_Z[\lambda+u]} &= \sum_{\substack{\lambda \in \mathbb{Z}^{2n} \\ \lambda_1 + u_1 \neq 0}} e^{-\pi\theta P_Z[\lambda+u]} + \sum_{\substack{\lambda \in \mathbb{Z}^{2n} \\ \lambda_1 + u_1 = 0 \\ \lambda_2 + u_2 \neq 0}} e^{-\pi\theta P_Z[\lambda+u]} \\ &\leq \sum_{\substack{\lambda_1, \lambda_2 \in \mathbb{Z}^n \\ \lambda_1 + u_1 \neq 0}} e^{-\pi\theta(e_{\min}\|\lambda_1+u_1\|_2^2 + e_{\max}^{-1}\|X(\lambda_1+u_1)+\lambda_2+u_2\|_2^2)} \\ &\qquad + \sum_{\substack{\lambda_1, \lambda_2 \in \mathbb{Z}^n \\ \lambda_1 + u_1 = 0 \\ \lambda_2 + u_2 \neq 0}} e^{-\pi\theta e_{\max}^{-1}\|\lambda_2+u_2\|_2^2} \\ &\leq C_{\theta_0}^n e^{-C(u_1)\pi\theta e_{\min}} (C_{\theta_0} - 1)^n + C_{\theta_0}^n e^{-C(u_2)\pi\theta e_{\max}^{-1}} \\ &\leq C_{\theta_0}^{2n} e^{-\min(C(u_1), C(u_2))\pi\theta a} \\ &\leq C_1(Y) e^{-C_2(u)\pi\theta a}, \end{aligned}$$

where

$$C_1(Y) = C_{\theta_0}^{2n} = \left(1 + 2\sum_{\lambda \geq 0} e^{-\pi a \lambda^2 \theta}\right)^{2n}$$

and

$$C_2(u) := \min\left(C(u_1), C(u_2)\right).$$

To establish an appropriate bound on $C_1(Y)$ now choose $A \in \mathbb{N}$ minimal such that $A \geq \frac{1}{a}$. Remember $\mathrm{tr}(Y) > 1$ since $Z \in \mathcal{F}_n$ and consider $\frac{1}{a} \leq \max(e_1^{-1}, e_2\mathrm{tr}(Y))$. It follows that A is bounded by a power of $\mathrm{tr}(Y)$. We estimate the inner sum of $C_1(Y)$.

$$\sum_{\lambda\geq 0} e^{-\pi a\lambda^2\theta} = \sum_{\substack{\mu\geq 0\\ 0\leq\nu<A}} e^{-\pi a(A\mu+\nu)^2\theta} \leq A\sum_{\mu\geq 0} e^{-\pi a(A\mu)^2\theta} \leq A\sum_{\mu\geq 0} e^{-\pi\mu^2\theta}.$$

So $C_1(Y) \leq (1+2A\sum_{\mu\geq 0} e^{-\pi\mu^2\theta})^{2n} \leq C_1'(Y)$ which is independent of θ is bounded by a power of $\mathrm{tr}(Y)$.

The integral of the expression (5) now reduces to

$$\left|\int_1^\infty \sum_{\substack{\lambda\in\mathbb{Z}^{2n}\\ \lambda+v\neq 0}} e^{-\pi\theta P_{M\langle Z\rangle}[\lambda+v]}\theta^{s-1}d\theta\right| \leq C_1'(Y)\int_1^\infty e^{-\pi\theta C_2(u)a}\theta^{C_s-1}d\theta,$$

where $C_s = \max\{\max_s(\mathfrak{Re}(s)), 1\}$.

We consider two cases. If $C_2(u)a \geq 1$ we get

$$\int_1^\infty e^{-\pi\theta C_2(u)a}\theta^{C_s-1}d\theta \leq \int_1^\infty e^{-\pi\theta}\theta^{C_s-1}d\theta,$$

which yields the desired result.

Lets now consider $C_2(u)a < 1$. Then we have

$$\begin{aligned}
&\int_1^\infty e^{-\pi\theta C_2(u)a}\theta^{C_s-1}d\theta\\
&\leq (C_2(u)a)^{-C_s}\left(\int_{C_2(u)a}^1 e^{-\pi\theta}\theta^{C_s-1}d\theta + \int_1^\infty e^{-\pi\theta}\theta^{C_s-1}d\theta\right)\\
&\leq (C_2(u)a)^{-C_s}\left(1+\int_1^\infty e^{-\pi\theta}\theta^{C_s-1}d\theta\right).
\end{aligned}$$

Since $a \geq \min(e_1, e_2^{-1}\mathrm{tr}(Y)^{-1})$ and $\mathrm{tr}(Y) > 1$ this yields the proposition's statement.■

—— *Functional equation* ——

We will establish the functional equation in several steps. First we deduce a sum in terms of P_Z.

Proposition (6.7)
Let χ_L be a primitive character with period L which induces χ. Furthermore let LR be the minimal period of χ and $\kappa \in \mathbb{N}$ such that $\kappa \mid N$. Then we have

$$\begin{aligned}
&\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z, 2-s) \\
&= \frac{\phi(R)}{R}\frac{\kappa}{G_{\overline{\chi_L}}}\sum_{r\,|\,R}\pi^{-s}\Gamma(s)(Lr)^{2s}\chi_L(r)\frac{\mu(r)}{\phi(r)}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}(\overline{\mathbb{1}_r\chi_L})(\lambda_4) \\
&\qquad \left(p^{\frac{3}{2}s}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix} Lrp\lambda_1 \\ Lr\kappa p\lambda_2 \\ Lrp\lambda_3 \\ \lambda_4\end{pmatrix}]^{-s} + p^{\frac{1}{2}s}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix} Lr\lambda_1 \\ Lr\kappa p\lambda_2 \\ Lr\lambda_3 \\ \lambda_4\end{pmatrix}]^{-s}\right). \ \square
\end{aligned}$$

Proof
Using (1.2) on page 6 and (6.4) on page 36 we get

$$\begin{aligned}
&\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z, 2-s) \\
&= N^{-2(2-s)}\sum_{\substack{\alpha(N),\,\beta(\kappa)\\ \gamma(N),\,\delta(N^2)}}\chi(\delta) \\
&\qquad\left(p^{\frac{1}{2}(2-s)-1}\sum_{h(p)}\zeta^*\left(s,\, -\left(\frac{\alpha}{N},\,\frac{p\beta}{\kappa},\,\frac{\gamma}{N},\,\frac{\delta}{N^2}\right)^{\mathrm{tr}},\,\left(0,\,\frac{\kappa h}{p},\,0,\,0\right)^{\mathrm{tr}},\,P_Z^{-1}\right)\right. \\
&\qquad\left.+p^{-\frac{1}{2}(2-s)}\sum_{g(p)}\zeta^*\left(s,\, -\left(\frac{\alpha}{N},\,\frac{\beta}{\kappa},\,\frac{\gamma}{N},\,\frac{p\delta+N^2g}{N^2p}\right)^{\mathrm{tr}},\,0,\,P_Z^{-1}\right)\right) \\
&= \pi^{-s}\Gamma(s)N^{-2(2-s)}\sum_{\substack{\alpha(N),\,\beta(\kappa)\\ \gamma(N),\,\delta(N^2)}}\chi(\delta) \\
&\qquad\left(p^{\frac{1}{2}(2-s)-1}\sum_{\substack{h(p)\\ v=\left(0,\,\frac{\kappa h}{p},\,0,\,0\right)^{\mathrm{tr}}}}\ \sum_{\substack{\lambda\in\mathbb{Z}^4\\ \lambda+v\neq 0}}\exp\left(-2\pi\imath\left(\frac{\alpha}{N},\,\frac{p\beta}{\kappa},\,\frac{\gamma}{N},\,\frac{\delta}{N^2}\right)\lambda\right)P_Z^{-1}[\lambda+v]^{-s}\right. \\
&\qquad\left.+p^{-\frac{1}{2}(2-s)}\sum_{g(p)}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\exp\left(-2\pi\imath\left(\frac{\alpha}{N},\,\frac{\beta}{\kappa},\,\frac{\gamma}{N},\,\frac{p\delta+N^2g}{N^2p}\right)\lambda\right)P_Z^{-1}[\lambda]^{-s}\right).
\end{aligned}$$

Let $\nu := \frac{N}{LR}$. If we sum over α, β, γ and δ and only consider the first exponential expression we get

$$\sum_{\substack{\alpha(N),\beta(\kappa)\\ \gamma(N),\delta(N^2)}} \chi(\delta)\exp\left(-2\pi\imath\left(\frac{\alpha}{N}, \frac{p\beta}{\kappa}, \frac{\gamma}{N}, \frac{\delta}{N^2}\right)\lambda\right)$$

$$= \begin{cases} N^3\nu\kappa\sum_{\delta(LR)}(\mathbb{1}_R\chi_L)(\delta)e^{-2\pi\imath\frac{\lambda_4}{N\nu}\frac{\delta}{LR}}, & \text{if } \begin{array}{l}\lambda_1\equiv\lambda_3\equiv 0\ (N),\\ \lambda_2\equiv 0\ (\kappa),\ \lambda_4\equiv 0\ (N\nu),\end{array}\\ 0, & \text{otherwise.}\end{cases}$$

By (4.1) on page 21 we may apply (4.2) on page 21 to $\chi = \mathbb{1}_R\chi_L$ we get

$$\sum_{\delta(LR)}(\mathbb{1}_R\chi_L)(\delta)e^{-2\pi\imath\frac{\lambda_4}{N\nu}\frac{\delta}{LR}} = G_{\chi_L}\chi_L(R)\overline{\chi_L}\left(-\frac{\lambda_4}{N\nu}\right)\mu(r)\phi\left(\frac{R}{r}\right),$$

where $r = \frac{R}{(R,\frac{\lambda_4}{N\nu})}$.

The reduction of the second sum can be done analogously. There occurs an additional factor $\sum_{g(p)} e^{-2\pi\imath g p^{-1}\lambda_4}$.

Putting all this together, using $G_{\chi_L}G_{\overline{\chi_L}} = \overline{\chi_L}(-1)L$ and replacing λ_1 by $N\lambda_1$, λ_2 by $\kappa\lambda_2$, λ_3 by $N\lambda_3$ and λ_4 by $N\nu\lambda_4$ we have

$$\left(\pi^{-s}\Gamma(s)\frac{\kappa}{RG_{\overline{\chi_L}}}\right)^{-1}\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z,2-s)$$

$$= N^{2s}\chi_L(R)\Bigg(p^{-\frac{1}{2}s}\sum_{\substack{\lambda\in\mathbb{Z}^4\\ h(p)\\ v=(N\lambda_1,\kappa\lambda_2,N\lambda_3,N\nu\lambda_4)\\ v+\left(0,\frac{\kappa h}{p},0,0\right)\neq 0}}\overline{\chi_L}(\lambda_4)\mu(r)\phi\left(\frac{R}{r}\right)P_Z^{-1}[\begin{pmatrix}N\lambda_1\\ \kappa\lambda_2+\frac{\kappa h}{p}\\ N\lambda_3\\ N\nu\lambda_4\end{pmatrix}]^{-s}$$

$$+ p^{\frac{1}{2}s-1}\sum_{\substack{\lambda\in\mathbb{Z}^4\setminus\{0\}\\ g(p)}}\overline{\chi_L}(\lambda_4)\mu(r)\phi\left(\frac{R}{r}\right)e^{-2\pi\imath\frac{g}{p}N\nu\lambda_4}P_Z^{-1}[\begin{pmatrix}N\lambda_1\\ \kappa\lambda_2\\ N\lambda_3\\ N\nu\lambda_4\end{pmatrix}]^{-s}\Bigg),$$

where $r = \frac{R}{(R,\lambda_4)}$.

Using (1.3) on page 6 and $JW_{Np}^{-1} = D_{Np}$ we get

$$P_Z^{-1} = P_{J\langle Z\rangle} = P_{JW_{Np}^{-1}W_{Np}\langle Z\rangle} = P_{W_{Np}\langle Z\rangle}\left[\left(JW_{Np}^{-1}\right)^{\mathrm{tr}}\right] = P_{W_{Np}\langle Z\rangle}\left[D_{Np}\right].$$

Now we fix one arbitrary r and consider the remaining sum.

$$N^{2s}\chi_L(R)\mu(r)\phi\left(\frac{R}{r}\right)\left(p^{-\frac{1}{2}s}\sum_{\substack{\lambda\in\mathbb{Z}^4\\ h(p)\\ v=(N\lambda_1,\kappa\lambda_2,N\lambda_3,N\nu\lambda_4)\\ v+\left(0,\frac{\kappa h}{p},0,0\right)\neq 0 (R,\lambda_4)=\frac{R}{r}}}\overline{\chi_L}(\lambda_4)P_{W_{Np}\langle Z\rangle}[\begin{pmatrix}N\lambda_1\\ N\kappa p\lambda_2+N\kappa h\\ N\lambda_3\\ \frac{\nu\lambda_4}{p}\end{pmatrix}]^{-s}\right.$$

$$\left.+p^{\frac{1}{2}s-1}\sum_{\substack{\lambda\in\mathbb{Z}^4\setminus\{0\}\\ g(p)\\ (R,\lambda_4)=\frac{R}{r}}}\overline{\chi_L}(\lambda_4)e^{-2\pi\imath\frac{g}{p}N\nu\lambda_4}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix}N\lambda_1\\ N\kappa p\lambda_2\\ N\lambda_3\\ \frac{\nu\lambda_4}{p}\end{pmatrix}]^{-s}\right).$$

We evaluate the sum over g and substitute λ_4 by $p\frac{R}{r}\lambda_4$ and $\frac{R}{r}\lambda_4$ respectively and substitute $p\lambda_2+h$ by λ_2. Recall that $p\equiv 1\ (N)$.

$$N^{2s}\chi_L(r)\mu(r)\phi\left(\frac{R}{r}\right)\left(p^{-\frac{1}{2}s}\sum_{\substack{\lambda\in\mathbb{Z}^4\setminus\{0\}\\ (r,\lambda_4)=1}}\overline{\chi_L}(\lambda_4)P_{W_{Np}\langle Z\rangle}[\begin{pmatrix}N\lambda_1\\ N\kappa\lambda_2\\ N\lambda_3\\ \nu\frac{R}{r}\frac{\lambda_4}{p}\end{pmatrix}]^{-s}\right.$$

$$\left.+p^{\frac{1}{2}s}\sum_{\substack{\lambda\in\mathbb{Z}^4\setminus\{0\}\\ (r,\lambda_4)=1}}\overline{\chi_L}(\lambda_4)P_{W_{Np}\langle Z\rangle}[\begin{pmatrix}N\lambda_1\\ N\kappa p\lambda_2\\ N\lambda_3\\ \nu\frac{R}{r}\lambda_4\end{pmatrix}]^{-s}\right)$$

$$=(Lr)^{2s}\chi_L(r)\mu(r)\phi\left(\frac{R}{r}\right)\left(p^{\frac{3}{2}s}\sum_{\substack{\lambda\in\mathbb{Z}^4\setminus\{0\}\\ (r,\lambda_4)=1}}\overline{\chi_L}(\lambda_4)P_{W_{Np}\langle Z\rangle}[\begin{pmatrix}Lrp\lambda_1\\ Lr\kappa p\lambda_2\\ Lrp\lambda_3\\ \lambda_4\end{pmatrix}]^{-s}\right.$$

$$\left.+p^{\frac{1}{2}s}\sum_{\substack{\lambda\in\mathbb{Z}^4\setminus\{0\}\\ (r,\lambda_4)=1}}\overline{\chi_L}(\lambda_4)P_{W_{Np}\langle Z\rangle}[\begin{pmatrix}Lr\lambda_1\\ Lr\kappa p\lambda_2\\ Lr\lambda_3\\ \lambda_4\end{pmatrix}]^{-s}\right)$$

$$= (Lr)^{2s}\chi_L(r)\mu(r)\phi\left(\frac{R}{r}\right)\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}(\overline{\mathbb{1}_r\chi_L})(\lambda_4)$$
$$\left(p^{\frac{3}{2}s}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix} Lrp\lambda_1 \\ Lr\kappa p\lambda_2 \\ Lrp\lambda_3 \\ \lambda_4\end{pmatrix}]^{-s} + p^{\frac{1}{2}s}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix} Lr\lambda_1 \\ Lr\kappa p\lambda_2 \\ Lr\lambda_3 \\ \lambda_4\end{pmatrix}]^{-s}\right).$$

Since R is square free we may use $\phi\left(\frac{R}{r}\right) = \frac{\phi(R)}{\phi(r)}$ and summing up the last expression for all $r \mid R$ yields the proposition's statement. ■

If we impose a further assumption on κ, we can deduce a more convenient expression.

Corollary (6.8)
Let χ_L be a primitive character with period L which induces χ. Furthermore let LR be the minimal period of χ and $\kappa \in \mathbb{N}$ such that $\kappa \mid L$. Then we have

$$\mathbb{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,\chi}(Z, 2-s) = \frac{\phi(R)}{R}\frac{\kappa}{G_{\overline{\chi_L}}}\sum_{r\mid R}\chi_L(r)\frac{\mu(r)}{\phi(r)}\mathbb{E}^{(p)*}_{Lrp,Lr\kappa p,\overline{\mathbb{1}_r\chi_L}}(W_{Np}\langle Z\rangle, s). \qquad \square$$

Proof
Apply (6.4) on page 36. ■

In general the Eisensteinseries occurring on the right hand side are hard to treat. By summing up smartly we can reduce them to one single function.

Proposition (6.9)
Let χ_L be a primitive character with period L which induces χ. Furthermore let LR be the minimal period of χ and $\kappa \in \mathbb{N}$ such that $\kappa \mid L$. Then we have

$$\sum_{\theta\mid R}\frac{\mu(\theta)\phi(\theta)}{\theta}\mathbb{E}^{(p)*}_{\frac{LR}{\theta}p,\frac{L^2R^2}{\kappa\theta^2}p,\overline{\mathbb{1}_{\frac{R}{r}}\chi_L}}(W_{\frac{LR}{\theta}p}\langle Z\rangle, 2-s)$$
$$= \frac{\mu(R)}{R}\frac{\chi_L(R)}{G_{\overline{\chi_L}}}\kappa\mathbb{E}^{(p)*}_{LRp,LR\kappa p,\overline{\mathbb{1}_R\chi_L}}(Z,\, s)\,. \qquad \square$$

Proof

The character $\mathbb{1}_{\frac{R}{\theta}}\chi_L$ has minimal period $\frac{LR}{\theta}$. Since $\kappa \mid L$ we may apply (6.8) to the left hand side. Exploiting the fact that R is square free we get

$$\begin{aligned}
&\sum_{\theta\,|\,R} \frac{\mu(\theta)\phi(\theta)}{\theta} \mathbb{E}^{(p)*}_{\frac{LR}{\theta}p,\frac{L^2R^2}{\kappa\theta^2}p,\overline{\mathbb{1}_{\frac{R}{r}}\chi_L}}(W_{\frac{LR}{\theta}p}\langle Z\rangle, 2-s)\\
&= \sum_{\theta\,|\,R} \frac{\mu(\theta)\phi(\theta)}{\theta} \frac{\phi\left(\frac{R}{\theta}\right)}{\frac{R}{\theta}} \frac{\kappa}{G_{\overline{\chi_L}}} \sum_{r\,|\,\frac{R}{\theta}} \chi_L(r)\frac{\mu(r)}{\phi(r)} \mathbb{E}^{(p)*}_{Lrp,\,L\kappa rp,\,\overline{\mathbb{1}_r\chi_L}}(Z,\,s)\\
&= \frac{\phi(R)}{R}\frac{\kappa}{G_{\overline{\chi_L}}} \sum_{\theta\,|\,R}\sum_{r\,|\,\frac{R}{\theta}} \mu(\theta)\chi_L(r)\frac{\mu(r)}{\phi(r)} \mathbb{E}^{(p)*}_{Lrp,\,L\kappa rp,\,\overline{\mathbb{1}_r\chi_L}}(Z,\,s)\\
&= \frac{\phi(R)}{R}\frac{\kappa}{G_{\overline{\chi_L}}} \sum_{r\,|\,R}\left(\sum_{\theta\,|\,\frac{R}{r}}\mu(\theta)\right)\chi_L(r)\frac{\mu(r)}{\phi(r)} \mathbb{E}^{(p)*}_{Lrp,\,L\kappa rp,\,\overline{\mathbb{1}_r\chi_L}}(Z,\,s)\\
&= \frac{\mu(R)}{R}\frac{\kappa}{G_{\overline{\chi_L}}}\chi_L(R)\mathbb{E}^{(p)*}_{LRp,\,LR\kappa p,\,\overline{\mathbb{1}_R\chi_L}}(Z,\,s)\,. \qquad\blacksquare
\end{aligned}$$

—— Reduction of index N ——

Until now we have succeeded in moving the major part of calculational complications to the left hand side in (6.9). We now develop a method for handling this. In this subsection we may again consider the weaker assumption $\kappa \mid N$.

Definition (6.10)

Let q be a prime and $r \in \mathbb{N}$ such that $(p, Nqr) = 1$. Then for $Z \in \mathbb{H}_2$, $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > 2$ we define $\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\,s)$ in terms of

$$\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\,s)\,|W_{Nqrp} = \mathbb{E}^{(p)*}_{Np,\,\frac{N^2}{\kappa}p,\,\chi}(\cdot,\,s)\,|W_{Np} - \mathbb{E}^{(p)*}_{Nqp,\,\frac{(Nq)^2}{\kappa}p,\,\mathbb{1}_q\chi}(\cdot,\,s)\,|W_{Nqp}\,. \quad\square$$

Proposition (6.11)

Let q be a prime and $r \in \mathbb{N}$ such that $(p, Nrq) = 1$. Then for $Z \in \mathbb{H}_2$, $s \in \mathbb{C}$ with $\mathfrak{Re}(s) > 2$ we have

(i) The considered function yields

$$\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(Z,\, s) = \pi^{-s}(Nrq)^{2s}\Gamma(s) \sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}} \chi(q\lambda_4)$$
$$\left(p^{\frac{1}{2}s}P_Z[\begin{pmatrix} Nqr\lambda_1 \\ \frac{N^2q^2r^2}{\kappa}p\lambda_2 \\ Nqr\lambda_3 \\ q\lambda_4 \end{pmatrix}]^{-s} + p^{\frac{3}{2}s}P_Z[\begin{pmatrix} Nqrp\lambda_1 \\ \frac{N^2q^2r^2}{\kappa}p\lambda_2 \\ Nqrp\lambda_3 \\ q\lambda_4 \end{pmatrix}]^{-s}\right).$$

(ii) If $q \mid N$ we have $\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(Z,\, s) = 0$.

(iii) $\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\, s) \in \left[\Gamma_{2,1}\left(Nqrp, (Nqrp)^2\right)\, 0,\, \overline{\chi}\right]$.

(iv) $\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\, s)\Big| M^{\mathrm{tr}}_{\frac{N^2r^2q}{\kappa}p} = \mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\, s)$. □

Proof

To proof (i) we regard (6.4) on page 36.

$$\begin{aligned}
&\left(\pi^{-s}\Gamma(s)\right)^{-1}\mathbb{E}^{(p)*}_{Np,\,\frac{N^2}{\kappa}p,\,\chi}(W_{Np}\langle Z\rangle,\, s) \\
&= N^{2s}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(\lambda_4)\left(p^{\frac{1}{2}s}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix} N\lambda_1 \\ \frac{N^2}{\kappa}p\lambda_2 \\ N\lambda_3 \\ \lambda_4 \end{pmatrix}]^{-s} + p^{\frac{3}{2}s}P_{W_{Np}\langle Z\rangle}[\begin{pmatrix} Np\lambda_1 \\ \frac{N^2}{\kappa}p\lambda_2 \\ Np\lambda_3 \\ \lambda_4 \end{pmatrix}]^{-s}\right) \\
&= (Nq)^{2s}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(\lambda_4) \\
&\qquad\left(p^{\frac{1}{2}s}P_{W_{Nqp}\langle Z\rangle}[\begin{pmatrix} Nq\lambda_1 \\ \frac{N^2q^2}{\kappa}p\lambda_2 \\ Nq\lambda_3 \\ \lambda_4 \end{pmatrix}]^{-s} + p^{\frac{3}{2}s}P_{W_{Nqp}\langle Z\rangle}[\begin{pmatrix} Nqp\lambda_1 \\ \frac{N^2q^2}{\kappa}p\lambda_2 \\ Nqp\lambda_3 \\ \lambda_4 \end{pmatrix}]^{-s}\right),
\end{aligned}$$

since $D_q^{-1}W_{Np} = W_{Npq}$. We split the addends with respect to $q \mid \lambda_4$ and $q \nmid \lambda_4$, respectively.

$$\begin{aligned}
&\left(\pi^{-s}\Gamma(s)\right)^{-1}\mathbb{E}^{(p)*}_{Np,\,\frac{N^2}{\kappa}p,\,\chi}\left(W_{Np}\langle Z\rangle,\,s\right)\\
&=\left(\pi^{-s}\Gamma(s)\right)^{-1}\mathbb{E}^{(p)*}_{Nqp,\,\frac{N^2q^2}{\kappa}p,\,\mathbb{1}_q\chi}\left(W_{Nqp}\langle Z\rangle,\,s\right)\\
&\quad+(Nq)^{2s}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(q\lambda_4)\\
&\qquad\left(p^{\frac{1}{2}s}P_{W_{Nqp}\langle Z\rangle}[\begin{pmatrix}Nq\lambda_1\\ \frac{N^2q^2}{\kappa}p\lambda_2\\ Nq\lambda_3\\ q\lambda_4\end{pmatrix}]^{-s}+p^{\frac{3}{2}s}P_{W_{Nqp}\langle Z\rangle}[\begin{pmatrix}Nqp\lambda_1\\ \frac{N^2q^2}{\kappa}p\lambda_2\\ Nqp\lambda_3\\ q\lambda_4\end{pmatrix}]^{-s}\right)\\
&=\left(\pi^{-s}\Gamma(s)\right)^{-1}\mathbb{E}^{(p)*}_{Nqp,\,\frac{N^2q^2}{\kappa}p,\,\mathbb{1}_q\chi}\left(W_{Nqp}\langle Z\rangle,\,s\right)\\
&\quad+(Nqr)^{2s}\sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}}\chi(q\lambda_4)\\
&\qquad\left(p^{\frac{1}{2}s}P_{W_{Nqrp}\langle Z\rangle}[\begin{pmatrix}Nqr\lambda_1\\ \frac{N^2q^2r^2}{\kappa}p\lambda_2\\ Nqr\lambda_3\\ q\lambda_4\end{pmatrix}]^{-s}+p^{\frac{3}{2}s}P_{W_{Nqrp}\langle Z\rangle}[\begin{pmatrix}Nqrp\lambda_1\\ \frac{N^2q^2r^2}{\kappa}p\lambda_2\\ Nqrp\lambda_3\\ q\lambda_4\end{pmatrix}]^{-s}\right).
\end{aligned}$$

This proofs (i) and (ii) is an immediate consequence since $\chi(q\lambda_4)=0$ if $q\mid N$. To proof (iii) we use the definition. We have

$$\begin{aligned}
&\mathbb{E}^{(p)*}_{Np,\,\frac{N^2}{\kappa}p,\,\chi}(\cdot,\,s)\,|W_{Np}\in\left[\Gamma_{2,1}(Np,N^2p),\,0,\,\chi\right]\qquad\text{and}\\
&\mathbb{E}^{(p)*}_{Nqp,\,\frac{(Nq)^2}{\kappa}p,\,\mathbb{1}_q\chi}(\cdot,\,s)\,|W_{Nqp}\in\left[\Gamma_{2,1}(Nqp,N^2q^2p),\,0,\,\chi\right].
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\,s)\,|W_{Nqrp}&\in\left[\Gamma_{2,1}(Nqp,N^2q^2p),\,0,\,\chi\right]\\
&\subseteq\left[\Gamma_{2,1}(Nqrp,(Nqrp)^2),\,0,\,\chi\right]\\
\Rightarrow\qquad\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\,s)&\in\left[\Gamma_{2,1}(Nqrp,(Nqrp)^2),\,0,\,\overline{\chi}\right].
\end{aligned}$$

There remains to prove (iv). This is an application of (i).

$$\mathcal{E}^{(p)*}_{Np,\frac{N^2}{\kappa}p,q,r,\chi}(\cdot,\,s)\,\Big|\,M^{\mathrm{tr}}_{\frac{N^2r^2q}{\kappa}p}$$

$$= \pi^{-s}(Nrq)^{2s}\Gamma(s) \sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}} \chi(q\lambda_4)$$

$$\left(p^{\frac{1}{2}s}P_Z[M_{\frac{N^2r^2q}{\kappa}p}\begin{pmatrix} Nqr\lambda_1 \\ \frac{N^2q^2r^2}{\kappa}p\lambda_2 \\ Nqr\lambda_3 \\ q\lambda_4 \end{pmatrix}]^{-s} + p^{\frac{3}{2}s}P_Z[M_{\frac{N^2r^2q}{\kappa}p}\begin{pmatrix} Nqrp\lambda_1 \\ \frac{N^2q^2r^2}{\kappa}p\lambda_2 \\ Nqrp\lambda_3 \\ q\lambda_4 \end{pmatrix}]^{-s}\right)$$

$$= \pi^{-s}(Nrq)^{2s}\Gamma(s) \sum_{\lambda\in\mathbb{Z}^4\setminus\{0\}} \chi(q\lambda_4)\left(p^{\frac{1}{2}s}P_Z[\begin{pmatrix} Nqr\lambda_1 \\ \frac{N^2q^2r^2}{\kappa}p\lambda_2 + \frac{N^2q^2r^2}{\kappa}p\lambda_4 \\ Nqr\lambda_3 \\ q\lambda_4 \end{pmatrix}]^{-s}\right.$$

$$\left. + p^{\frac{3}{2}s}P_Z[\begin{pmatrix} Nqrp\lambda_1 \\ \frac{N^2q^2r^2}{\kappa}p\lambda_2 + \frac{N^2q^2r^2}{\kappa}p\lambda_4 \\ Nqrp\lambda_3 \\ q\lambda_4 \end{pmatrix}]^{-s}\right).$$

If we fix any λ_4, the second entry of both vectors runs through all integers divisible by $\frac{N^2q^2r^2p}{\kappa}$. Thus the last statement is valid. ■

§ 7 First integral representation of $\mathbb{D}_{f,g,\chi}$

Using the Eisensteinseries in 6 we are able to deduce a representation of $\mathbb{D}_{f,g,\chi}$. The exceptional case $p = 1$ has been treated by Kuß in [5]. We use the same conventions as in section 5. Let p be a prime, χ be a character of period N, where $p \equiv 1\ (N)$. Let $f, g \in [\Gamma_{2,1}^{(p)*}(p,p), k, \mathbb{1}_1^{k-}]_0$ be a Siegel cusp form with Fourier-Jacobi series $f(Z) = \sum_{m=1}^{\infty} f_m(Z_1, w) e^{2\pi\imath mz}$ and $g(Z) = \sum_{m=1}^{\infty} g_m(Z_1, w) e^{2\pi\imath mz}$.

Theorem (7.1)
Let $\chi^{(2)}$ be a primitive character with periode $L^{(2)}$ which induces χ^2. Let θ be a natural such that $\theta \mid \frac{N}{L^{(2)}}$. If $\theta \neq 1$ let χ be primitive. Then we have

$$\begin{aligned}&\left\{ f_\chi \,|W_{Np}\, \mathbb{E}^{(p)*}_{\frac{N}{\theta}p,\, \frac{N^2}{\theta^2}p,\, \mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}}(\cdot, s) \,\Big|W_{\frac{N}{\theta}p}\,,\, g\,|W_{Np} \right\}\\ &= \frac{2}{i_N}\left(\frac{\pi}{N^2}\right)^{k-2} p^{\frac{3}{2}s-1}(1+p^{-s})\mathbb{D}_{f,\,g,\,\chi}(s+k-2).\end{aligned}$$

□

Proof
The first and second argument have the common symmetry group $\Gamma_{2,1}(Np, (Np)^2)$. Since W_{Np} normalises this group, we see that the scalar product converges absolutely if we consider (6.6) on page 40 and

$$\begin{aligned}&\left\{ f_\chi \,|W_{Np}\, \mathbb{E}^{(p)*}_{\frac{N}{\theta}p,\, \frac{N^2}{\theta^2}p,\, \mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}} \,\Big|W_{\frac{N}{\theta}p}\,(\cdot, s),\, g\,|W_{Np} \right\}\\ &= \left\{ f_\chi \mathbb{E}^{(p)*}_{\frac{N}{\theta}p,\, \frac{N^2}{\theta^2}p,\, \mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}}(\cdot, s),\, g \right\}.\end{aligned}$$

Now we use induction on θ. There for let $\theta = 1$. We continue the calculation and apply the unfolding trick.

$$\begin{aligned}H(s) &:= \left(\frac{2}{i_N}\pi^{-s}N^{2s}p^{\frac{3}{2}s}(1+p^{-s})\Gamma(s)L(2s,\chi^2)\right)^{-1} \left\{ f_\chi \mathbb{E}^{(p)*}_{\frac{N}{\theta}p,\, \frac{N^2}{\theta^2}p,\, \mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}}(\cdot, s),\, g \right\}\\ &= \left(\frac{2}{i_N}\right)^{-1} \left\{ f_\chi E^{(p)*}_{\frac{N}{\theta}p,\, \frac{N^2}{\theta^2}p,\, \mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}}(\cdot, s),\, g \right\}\\ &= \int_{\mathcal{F}_{2,1}^{(p)}} f_\chi(Z)\overline{g(Z)}(\det Y_1)^{k-3}\left(y - Y_1^{-1}[v]\right)^{s+k-3} dX dY\end{aligned}$$

Here we see by standard estimates for Siegel modular forms, that the integral over the Fourier series converges absolutely and hence later we may reorder the series.

$$\begin{aligned}
H(s) :=& \int\limits_{\substack{(Z_1,\,w)\in\mathcal{F}_2^*\\ y>Y_1^{-1}[v]\\ x\in[0,\frac{1}{p}]}} \sum_{m_f,\,m_g=1}^{\infty} \chi(m_f) f_{pm_f}(Z_1,\,w)\overline{g_{pm_g}(Z_1,\,w)}\\
&\qquad \cdot e^{-2\pi(m_f+m_g)py+2\pi\imath(m_f-m_g)px}\\
&\qquad \cdot (\det Y_1)^{k-3}(y-Y_1^{-1}[v])^{s+k-3}dxdydudvdX_1dY_1\\
=& p^{-1}\int\limits_{\substack{(Z_1,\,w)\in\mathcal{F}_2^*\\ y>Y_1^{-1}[v]}} \sum_{m=1}^{\infty}\chi(m) f_{pm}(Z_1,\,w)\overline{g_{pm}(Z_1,\,w)}\\
&\qquad \cdot e^{-4\pi pmy}(\det Y_1)^{k-3}(y-Y_1^{-1}[v])^{s+k-3}dydudvdX_1dY_1\\
=& p^{-1}(4\pi)^{-s-k+2}\Gamma(s+k-2)\\
&\int\limits_{(Z_1,\,w)\in\mathcal{F}_2^*}\sum_{m=1}^{\infty}\chi(m)(pm)^{-s-k+2} f_{pm}(Z_1,\,w)\overline{g_{pm}(Z_1,\,w)}\\
&\qquad \cdot e^{-4\pi pmY_1^{-1}[v]}(\det Y_1)^{k-3}dudvdX_1dY_1\\
=& p^{-1}(4\pi)^{-s-k+2}\Gamma(s+k-2)\sum_{m=1}^{\infty}\chi(m)\left\langle f_{pm},\,g_{pm}\right\rangle (pm)^{-s-k+2}\\
=& p^{-1}(4\pi)^{-s-k+2}\Gamma(s+k-2)D_{f,\,g,\,\chi}(s+k-2).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\{ f_\chi \mathbb{E}^{(p)*}_{\frac{N}{\theta}p,\,\frac{N^2}{\theta^2}p,\,\mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}}(\cdot,s),\,g\right\}\\
&= \frac{2}{i_N}\pi^{-s}N^{2s}p^{\frac{3}{2}s}(1+p^{-s})\Gamma(s)L(2s,\chi^2)p^{-1}(4\pi)^{-s-k+2}\Gamma(s+k-2)D_{f,\,g,\,\chi}(s+k-2)\\
&= \frac{2}{i_N}\left(\frac{\pi}{N^2}\right)^{k-2}p^{\frac{3}{2}s-1}(1+p^{-s})\mathbb{D}_{f,\,g,\,\chi}(s+k-2).
\end{aligned}$$

This proves the statement if $\theta = 1$.

Now assume $\theta \neq 1$ and hence there is a prime q such that $q \mid \theta$. We have

$$\begin{aligned}
\mathbb{E}^{(p)*}_{\frac{N}{\theta}p,\,\frac{N^2}{\theta^2}p,\,\chi^2}(\cdot,\,s)\Big|W_{\frac{N}{\theta}p} =& \mathbb{E}^{(p)*}_{\frac{N}{\theta/q}p,\,\frac{N^2}{(\theta/q)^2}p,\,\chi^2}(\cdot,\,s)\Big|W_{\frac{N}{\theta/q}p}\\
&+\mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s)\,|W_{Np}\;.
\end{aligned}$$

Thus by induction it suffices to show that

$$\left\{ f_\chi \,|W_{Np}\; \mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s)\,|W_{Np}\;,\; g\,|W_{Np} \right\} = \left\{ f_\chi \mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s),\; g \right\}$$

vanishes. By (ii) in (6.11) on page 47 we see that this is true if $q \mid \frac{N}{\theta}$. Hence we may assume $q \nmid \frac{N}{\theta}$.

The modular forms in the last equation have a common symmetry group $\Gamma^1_{2,1}\left(Np,\,(Np)^2\right)$ and M^{tr}_{Np} normalises this group as shown in (3.4) on page 14 hence so does $M^{\mathrm{tr}}_{(Np)^2\frac{\nu}{q}}$ if $\nu \in \mathbb{N}$. We have

$$\left\{ f_\chi \mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s),\; g \right\} = \frac{1}{q}\sum_{\nu(q)} \left\{ f_\chi \left| M^{\mathrm{tr}}_{(Np)^2\frac{\nu}{q}} \right. \mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s) \left| M^{\mathrm{tr}}_{(Np)^2\frac{\nu}{q}} \right.,\; g \left| M^{\mathrm{tr}}_{(Np)^2\frac{\nu}{q}} \right. \right\}.$$

We see $M^{\mathrm{tr}}_{(Np)^2\frac{\nu}{q}} \in \Gamma^{(p)}_{2,1}(p,p)$ and using (iv) in (6.11) on page 47 we continue

$$\left\{ f_\chi \mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s),\; g \right\} = \frac{1}{q}\sum_{\nu(q)} \left\{ f_\chi \left| M^{\mathrm{tr}}_{(Np)^2\frac{\nu}{q}} \right. \mathcal{E}^{(p)*}_{\frac{N}{\theta}p,\frac{N^2}{\theta^2}p,q,\frac{\theta}{q},\chi^2}(\cdot,\,s),\; g \right\}.$$

This sum vanishes by (5.6) on page 32. ■

Using this theorem we are able to proof our first important result.

Corollary (7.2)
Suppose that χ is primitive. Let $\chi^{(2)}$ be a primitive character with period $L^{(2)}$ which induces χ^2. Let $L^{(2)}R^{(2)}$ be the minimal period of χ^2. Then we have

$$\left\{ f_\chi \,|W_{Np} \sum_{\theta \,|\, R^{(2)}} \frac{\mu(\theta)\phi(\theta)}{\theta} \mathbb{E}^{(p)*}_{\frac{L^{(2)}R^{(2)}}{\theta}p,\left(\frac{L^{(2)}R^{(2)}}{\theta}\right)^2 p,\mathbb{1}_{\frac{N}{\theta}}\chi^{(2)}}(\cdot, s) \left| W_{\frac{L^{(2)}R^{(2)}}{\theta}} \right.,\; g\,|W_{Np} \right\} = \frac{2}{i_N R^{(2)}}\left(\frac{\pi}{N^2}\right)^{k-2} p^{\frac{3}{2}s-1}(1+p^{-s})\mathbb{D}_{f,g,\chi}(s+k-2).$$

□

Proof

Using theorem (7.1) we just have to calculate

$$\sum_{\theta \mid R^{(2)}} \frac{\mu(\theta)\phi(\theta)}{\theta} = \prod_{p \mid R^{(2)}} \left(\sum_{\theta \mid p} \frac{\mu(\theta)\phi(\theta)}{\theta} \right) = \prod_{p \mid R^{(2)}} \left(1 + \frac{-1(p-1)}{p} \right)$$
$$= \prod_{p \mid R^{(2)}} \frac{1}{p} = \frac{1}{R^{(2)}}. \qquad \blacksquare$$

§ 8 Second integral representation

In this section we will prove a second integral representation associated to (7.2) on page 53 by means of (6.9) on page 46. The ideas for this approach where first developed by Kohnen, Krieg and Sengupta in [9] and strongly refined by Kuß in [5].

We adopt the conventions of section 7.

Let p be a prime, χ be a character of period N, where $p \equiv 1\ (N)$. Let $\chi^{(2)}$ be a primitive character which induces χ^2 and let $L^{(2)}R^{(2)}$ be the minimal period of χ^2. Set $\nu := \frac{N}{L^{(2)}R^{(2)}}$. Let $f, g \in [\Gamma_{2,1}^{(p)*}(p,p), k, \mathbb{1}_1^{k-}]_0$ be a Siegel cusp form with Fourier-Jacobi series $f(Z) = \sum_{m=1}^{\infty} f_m(Z_1, w)e^{2\pi\imath mz}$ and $g(Z) = \sum_{m=1}^{\infty} g_m(Z_1, w)e^{2\pi\imath mz}$.

We first calculate some Fourier-Jacobi series.

Lemma (8.1)

Suppose $f, g \in [\Gamma_{2,1}(p,p), k, \mathbb{1}_1]_0$ with Fourier-Jacobi series as given above.

Suppose $d \mid \nu$ and $\lambda \in \mathbb{Z}^{2\times}$. Let $C \mid N\nu$, $\gamma \in \mathbb{N}$ such that $(\gamma, \frac{N\nu}{C}) = 1$ and let $\epsilon \in \{\pm 1\}$. Choose $\gamma^* \in \mathbb{N}$ by means of $\gamma\gamma^* \equiv 1\ \left(\frac{N\nu}{C}\right)$, choose p^* by means of $pp^* \equiv 1\ \left(\frac{N\nu}{C}\right)$ and a minimal $\mu \mid \nu$ such that $C \mid Nd\mu$. Choose $\mu^* \in \mathbb{N}$ such that $\mu\mu^* \equiv 1\ \left(\frac{Nd\mu p}{C}\right)$.

If $M_{d,C\gamma}$ is chosen as in (3.10) on page 18 with $\theta = \frac{N}{\nu}$ and M_λ as on page 13, for any $\epsilon_\gamma \equiv \gamma\ (\nu)$ we have a Fourier-Jacobi series

$$f\,|W_{N\nu p}M_{d,C\gamma}M_\lambda D_\epsilon = \sum_{m=1}^{\infty} f_{m,C,d,\lambda,\epsilon,\epsilon_\gamma}(Z_1, w)e^{2\pi\imath\frac{C^2}{\mu^2}mz}e^{2\pi\imath\frac{C\mu^{*2}\gamma^*}{N\nu}p^*m},$$

where $f_{m,C,d,\lambda,\epsilon_\gamma}$ depends on γ and γ^* only by means of ϵ_γ and $f_{m,C,d,\lambda,\epsilon_\gamma} \equiv 0$ if $p \nmid m$.

Suppose $\mu = \mu^* = 1$ and $C\nu \mid N$. Then we have

$$f_{m,C,d,\lambda,\epsilon,\epsilon_\gamma}(Z_1, w) = C^k f_m\left(Z_1, Cw + \frac{pp^*\epsilon\epsilon_\gamma}{\nu}(Z_1 d\lambda_1 + d\lambda_2)\right)$$
$$e^{2\pi\imath m\left(\frac{pp^*\epsilon\epsilon_\gamma}{\nu}\right)^2(Z_1 d^2\lambda_1^2 + d^2\lambda_1\lambda_2)}e^{2\pi\imath m\frac{2Cpp^*\epsilon\epsilon_\gamma}{\nu}wd\lambda_1}. \qquad \square$$

Proof

We consider $W_{N\nu p}M_{d,C\gamma}$ and transform it by $-W_p \in \Gamma_{2,1}^{(p)}(p,p)$. There remains

$$M_1 := -W_pW_{N\nu p}M_{d,C\gamma} = D_{N\nu}M_{d,C\gamma} = \left(\begin{array}{cc|cc} 1 & -Ndp & & \\ & N\nu & & \\ \hline & & 1 & \\ & pC\gamma & \frac{dp}{\nu} & \frac{1}{N\nu} \end{array}\right).$$

We consider

$$H_1 := \left(\begin{array}{cc|cc} 1 & & & \\ & \frac{(1-p\gamma p^*\gamma^*)C}{N\nu} & & p^*\gamma^* \\ \hline & & 1 & \\ & -p\gamma & & \frac{N\nu}{C} \end{array}\right)$$

$$M_2 := H_1 M_1 = \left(\begin{array}{cc|cc} 1 & -Ndp & & \\ & C & \frac{dpp^*\gamma^*}{\nu} & \frac{p^*\gamma^*}{N\nu} \\ \hline & & 1 & \\ & & \frac{Ndp}{C} & \frac{1}{C} \end{array}\right).$$

By definition of γ^* and p^* we have $H_1 \in \Gamma_{2,1}(p,p)$.

$$H_2 := \left(\begin{array}{cc|cc} \mu & \frac{Ndp\mu}{C} & & \\ \frac{(\mu\mu^*-1)C}{Ndp\mu} & \mu^* & & \\ \hline & & \mu^* & \frac{(1-\mu\mu^*)C}{Ndp\mu} \\ & & -\frac{Ndp\mu}{C} & \mu \end{array}\right)$$

$$M_3 := H_2 M_2 = \left(\begin{array}{cc|cc} \mu & & \frac{Nd^2p^2p^*\gamma^*\mu}{C\nu} & \frac{dpp^*\gamma^*\mu}{C\nu} \\ \frac{(\mu\mu^*-1)C}{Ndp\mu} & \frac{C}{\mu} & \frac{dpp^*\gamma^*\mu^*}{\nu} & \frac{p^*\gamma^*\mu^*}{N\nu} \\ \hline & & \frac{1}{\mu} & \frac{1-\mu\mu^*}{Ndp\mu} \\ & & & \frac{\mu}{C} \end{array}\right).$$

By the assumptions on ϵ_γ we have $r := \frac{\epsilon_\gamma - \gamma^*}{\nu} \in \mathbb{Z}$ and hence the following matrix has integral entries.

$$H_3 := \left(\begin{array}{cc|cc} 1 & & \frac{Nd^2p^2p^*\mu^2}{C}r & dpp^*\mu^*\mu r \\ & 1 & dpp^*\mu^*\mu r & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

$$M_4 := H_3 M_3 = \left(\begin{array}{cc|cc} \mu & & \frac{Nd^2p^2p^*\mu\epsilon_\gamma}{C\nu} & \frac{dpp^*\mu\epsilon_\gamma}{C\nu} \\ \frac{(\mu\mu^*-1)C}{Ndp\mu} & \frac{C}{\mu} & \frac{dpp^*\mu^*\epsilon_\gamma}{\nu} & \frac{p^*\mu^*(\epsilon_\gamma-\mu\mu^*\nu r)}{N\nu} \\ \hline & & \frac{1}{\mu} & \frac{1-\mu\mu^*}{Ndp\mu} \\ & & & \frac{\mu}{C} \end{array}\right).$$

The first statement follows since multiplication with M_λ from the right won't change the given exponential terms of the according substitution and D_ϵ yields multiplication of w by ϵ.

Now suppose $\mu = 1$ and $C\nu \mid N$. Then we may chose $\mu^* = 1$ and the last expression simplifies.

$$M_4 = \left(\begin{array}{cc|cc} 1 & & \frac{Nd^2p^2p^*\epsilon_\gamma}{C\nu} & \frac{dpp^*\epsilon_\gamma}{C\nu} \\ & C & \frac{dpp^*\epsilon_\gamma}{\nu} & \frac{p^*\gamma^*}{N\nu} \\ \hline & & 1 & \\ & & & \frac{1}{C} \end{array}\right).$$

We calculate

$$H_4 := \left(\begin{array}{cc|cc} 1 & & \frac{Nd^2p^2p^*\epsilon_\gamma}{C\nu} & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$$

$$M_5 := H_4M_4 = \left(\begin{array}{cc|cc} 1 & & & \frac{dpp^*\epsilon_\gamma}{C\nu} \\ & C & \frac{dpp^*\epsilon_\gamma}{\nu} & \frac{p^*\gamma^*}{N\nu} \\ \hline & & 1 & \\ & & & \frac{1}{C} \end{array}\right).$$

We regard M_λ and see

$$M_\lambda^{-1}M_5M_\lambda = \left(\begin{array}{cc|cc} 1 & & & \frac{dpp^*\epsilon_\gamma}{C\nu}\lambda_2 \\ \frac{dpp^*\epsilon_\gamma}{\nu}\lambda_1 & C & \frac{dpp^*\epsilon_\gamma}{\nu}\lambda_2 & \frac{p^*\gamma^*}{N\nu} \\ \hline & & 1 & -\frac{dpp^*\epsilon_\gamma}{C\nu}\lambda_1 \\ & & & \frac{1}{C} \end{array}\right).$$

Conjugation by D_ϵ yields the stated Fourier-Jacobi series.

$$D_\epsilon M_\lambda^{-1}M_5M_\lambda D_\epsilon = \left(\begin{array}{cc|cc} 1 & & & \frac{dpp^*\epsilon\epsilon_\gamma}{C\nu}\lambda_2 \\ \frac{dpp^*\epsilon\epsilon_\gamma}{\nu}\lambda_1 & C & \frac{dpp^*\epsilon\epsilon_\gamma}{\nu}\lambda_2 & \frac{p^*\gamma^*}{N\nu} \\ \hline & & 1 & -\frac{dpp^*\epsilon\epsilon_\gamma}{C\nu}\lambda_1 \\ & & & \frac{1}{C} \end{array}\right). \qquad \blacksquare$$

Using this lemma we are able to deduce

Theorem (8.2)
We have

$$\left\{ f_\chi \,|W_{Np}\; \mathbb{E}^{(p)*}_{\frac{N}{\nu}p,\,\frac{N}{\nu}p,\,\overline{\chi}^2}(\cdot, s),\ g\,|W_{Np} \right\}$$
$$= \mu(R)\overline{\chi}^{(2)}(R)G_\chi^4 G_{\overline{\chi}^{(2)}} p^{\frac{3}{2}s-1}(1+p^{-s})\frac{2}{i_N N^2}\left(\frac{\pi}{N^2}\right)^{k-2}\mathbb{D}_{f,g,\overline{\chi}}(s+k-2). \qquad \square$$

Proof

The integral is well defined and converges absolutely if we consider (6.6) on page 40. Thus we may calculate using (ii) of (6.11) on page 47 and considering the definition of $\{\cdot,\cdot\}$

$$\begin{aligned}
&H(s)\\
&:= \left\{ f_\chi\,|W_{Np}\,\mathbb{E}^{(p)*}_{\frac{N}{\nu}p,\,\frac{N}{\nu}p,\,\overline{\chi}^2}(\cdot,s),\ g\,|W_{Np} \right\}\\
&= \left\{ f_\chi \mathbb{E}^{(p)*}_{\frac{N}{\nu}p,\,\frac{N}{\nu}p,\,\overline{\chi}^2}(\cdot,s)\,|W_{Np}\ ,\ g \right\}\\
&= \left\{ f_\chi \mathbb{E}^{(p)*}_{Np,\,N\nu p,\,\overline{\chi}^2}(\cdot,s)\,|W_{N\nu p}\ ,\ g \right\}\\
&= \left\{ f_\chi\,|W_{N\nu p}\,\mathbb{E}^{(p)*}_{Np,\,N\nu p,\,\overline{\chi}^2}(\cdot,s),\ g\,|W_{N\nu p} \right\}\\
&= \left\{ \mathbb{E}^{(p)*}_{Np,\,N\nu p,\,\overline{\chi}^2}(\cdot,s),\ \overline{f_\chi\,|W_{N\nu p}}\, g\,|W_{N\nu p}\ (\det \mathfrak{Im}(\cdot))^k \right\}\\
&= \left\{ \mathbb{E}^{(p)*}_{Np,\,N\nu p,\,\overline{\chi}^2}(\cdot,s),\ \operatorname{trace}\left(\overline{f_\chi\,|W_{N\nu p}}\, g\,|W_{N\nu p}\ (\det \mathfrak{Im}(\cdot))^k \right) \right\}.
\end{aligned}$$

Here we use the trace operator

$$\begin{aligned}
\operatorname{trace}\ :&[\Gamma^{(p)*}_{2,1}(N\nu p,\,(N\nu)^2 p),\,0,\,\left(\overline{\chi}^2\right)^+] \longrightarrow [\Gamma^{(p)*}_{2,1}(Np,\,N\nu p),\,0,\,\left(\overline{\chi}^2\right)^+]\\
&f \mapsto \frac{1}{c} \sum_{M:\Gamma^{(p)*}_{2,1}(N\nu p,\,(N\nu)^2 p)\backslash\Gamma^{(p)*}_{2,1}(Np,\,N\nu p)} \left(\overline{\chi}^2\right)^+(M) f\,|M\ ,
\end{aligned}$$

where

$$c = [\Gamma^{(p)*}_{2,1}(Np,\,N\nu p)\ :\ \Gamma^{(p)*}_{2,1}(N\nu p,\,(N\nu)^2 p)] = N\nu^3.$$

Recall, by (5.5) on page 31, f_χ as well as g have the correct symmetry group.

We use the system of representatives given in (3.12) on page 19 where $\theta = \frac{N}{\nu}$ and adopt the notation of $M_{d,\gamma}$. We now consider

$$\begin{aligned}
W^{-1}_{N\nu p} M_{\frac{\mu}{N}} W_{N\nu p} &= M^{\mathrm{tr}}_{-N\nu^2 p^2 \mu}\\
M^{\mathrm{tr}}_{-N\nu^2 p^2 \mu} M_{d,\beta} &= M_{d,\beta-\nu p\mu},
\end{aligned}$$

and define

$$\begin{aligned}
&\operatorname{trace}\left(\overline{f_\chi\,|W_{N\nu p}}\, g\,|W_{N\nu p}\ (\det \mathfrak{Im}(\cdot))^k \right)\\
&=: \frac{1}{c}\overline{S}\cdot(\det \mathfrak{Im}(\cdot))^k
\end{aligned}$$

with

$$
\begin{aligned}
G_{\overline{\chi}} S &= G_{\overline{\chi}} \sum_{\substack{d\,|\,\nu \\ \lambda\in\{1,\dots,\frac{\nu}{d}\}^{2\times} \\ \beta(N\nu)}} f_\chi\,|W_{N\nu p} M_{d,\beta} M_\lambda\, \overline{g\,|W_{N\nu p} M_{d,\beta} M_\lambda} \\
&= \sum_{\substack{d\,|\,\nu \\ \lambda\in\{1,\dots,\frac{\nu}{d}\}^{2\times} \\ \beta(N\nu) \\ \mu(N)}} \overline{\chi}(\mu) f\,\Big|M_{\frac{\mu}{N}} W_{N\nu p} M_{d,\beta} M_\lambda\, \overline{g\,|W_{N\nu p} M_{d,\beta} M_\lambda} \\
&= \sum_{\substack{d\,|\,\nu \\ \lambda\in\{1,\dots,\frac{\nu}{d}\}^{2\times} \\ \beta(N\nu) \\ \mu(N)}} \overline{\chi}(\mu) f\,|W_{N\nu p} M_{d,\beta-\nu p\mu} M_\lambda\, \overline{g\,|W_{N\nu p} M_{d,\beta} M_\lambda} \\
&= \sum_{\substack{d\,|\,\nu \\ \lambda\in\{1,\dots,\frac{\nu}{d}\}^{2\times} \\ \beta(N\nu) \\ \mu(N)}} \overline{\chi}(\mu) f\,|W_{N\nu p} M_{d,\beta-\nu\mu} M_\lambda\, \overline{g\,|W_{N\nu p} M_{d,\beta} M_\lambda} \\
&= \sum_{\substack{d\,|\,\nu \\ \lambda\in\{1,\dots,\frac{\nu}{d}\}^{2\times} \\ \beta(N\nu) \\ \gamma(N\nu) \\ \beta\equiv\gamma(\nu)}} \overline{\chi}\left(\frac{\beta-\gamma}{\nu}\right) f\,|W_{N\nu p} M_{d,\gamma} M_\lambda\, \overline{g\,|W_{N\nu p} M_{d,\beta} M_\lambda} \\
&= \sum_{\substack{d\,|\,\nu \\ \lambda\in\{1,\dots,\frac{\nu}{d}\}^{2\times} \\ B\,|\,N\nu,\,\beta\left(\frac{N\nu}{B}\right)^\times \\ C\,|\,N\nu,\,\gamma\left(\frac{N\nu}{C}\right)^\times \\ B\beta\equiv C\gamma(\nu)}} \overline{\chi}\left(\frac{B\beta-C\gamma}{\nu}\right) f\,|W_{N\nu p} M_{d,C\gamma} M_\lambda\, \overline{g\,|W_{N\nu p} M_{d,B\beta} M_\lambda}.
\end{aligned}
$$

We consider the scalar product and apply the unfolding trick. Mind that S is invariant if we apply D_ϵ, where $\epsilon \in \{\pm 1\}$. We sum the fundamental domains $\mathcal{F}_{2,1}^{(p)}$ and $D_{-1}\mathcal{F}_{2,1}^{(p)}$.

$$
\begin{aligned}
&\left(\pi^{-s} N^{2s} p^{\frac{3}{2}s}(1+p^{-s})\Gamma(s) L(2s,\chi^2) G_{\overline{\chi}}^{-1}\right)^{-1} H(s) \\
&= \left(\pi^{-s} N^{2s} p^{\frac{3}{2}s}(1+p^{-s})\Gamma(s) L(2s,\chi^2) G_{\overline{\chi}}^{-1}\right)^{-1} \\
&\quad \cdot \left\{\mathbb{E}^{(p)*}_{Np,\,N\nu p,\,\overline{\chi}^2}(\cdot,s),\ \mathrm{trace}\left(\overline{f_\chi\,|W_{N\nu p}}\, g\,|W_{N\nu p}\,(\det \mathfrak{Im}(\cdot))^k\right)\right\}
\end{aligned}
$$

$$= \frac{1}{[\Gamma_{2,1}^{(p)*}(p,p) \,:\, \Gamma_{2,1}^{(p)*}(Np, N\nu p)]c}$$

$$\int\limits_{\mathcal{F}_{2,1}^{(p)} \cup D_{-1}\mathcal{F}_{2,1}^{(p)}} (\det Y_1)^{k-3}(y - Y_1^{-1}[v])^{s+k-3} G_{\overline{\chi}} S(Z) dX dY$$

$$= \frac{1}{2 i_N \frac{\nu}{N} N \nu^3}$$

$$\int\limits_{\substack{Z_1 \in \mathcal{F}_1 \\ v \in \mathbb{R}/Y_1\mathbb{Z} \\ u \in [-\frac{1}{2}, \frac{1}{2}] \\ y > Y_1^{-1}[v] \\ x \in [0, \frac{1}{p}]}} (\det Y_1)^{k-3}(y - Y_1^{-1}[v])^{s+k-3} \sum_{\epsilon \in \{\pm 1\}} G_{\overline{\chi}} S \,|D_\epsilon (Z) dX dY.$$

We start showing that all "bad" addends of S vanish.

We apply lemma (8.1) on page 55 to f and g and adopt the notation of $\mu^{(B)}, \mu^{(C)}$ as well as m_B, β^* and ϵ_β and m_C etc.. We fix B, C, d and λ and choose ϵ_γ and ϵ_β to be 1 if $\nu \neq 4$ and otherwise to be either 1 or -1.

Considering the integration over x we get the expression

$$\int\limits_{x \in [0, \frac{1}{p}]} e^{2\pi\imath x \left(\frac{C^2}{\mu^{(C)^2}} m_C - \frac{B^2}{\mu^{(B)^2}} m_B \right)}.$$

If this doesn't vanish we have $p \mid m_B, m_C$ and hence

$$\frac{C^2}{\mu^{(C)^2}} m_C = \frac{B^2}{\mu^{(B)^2}} m_B. \tag{6}$$

Assume $\mu^{(B)} \neq 1$ or $\mu^{(C)} \neq 1$. For reasons of symmetry we may suppose that $\mu^{(C)} \neq 1$.

We then find $2\mu^{(C)}\nu \mid C$. Indeed, since $\nu \neq 1$ follows from $\mu^{(C)} \neq 1$ we have $2\nu \mid N$ by (4.6) on page 24. Then from $\nu_2(C) = \nu_2(Ndp\mu^{(C)})$ it follows $2\mu^{(C)}\nu \mid C$.

So instead of $\mu^{(C)} \neq 1$ we may assume $2\mu^{(C)}\nu \mid C$. Then by $2\nu \mid C$ and $C \mid N\nu$ we find $2 \mid N$ and hence $4 \mid N$, since χ is primitive. By $\beta B \equiv \gamma C \ (\nu)$ we see $\nu \mid B$ and on the other hand we may assume $\nu_2(B) = \nu_2(\nu)$ since otherwise $\overline{\chi}\left(\frac{B\beta - C\gamma}{\nu}\right)$ already vanishes. Since $2\nu \mid N$ we then have $B \mid N$ and conclude $\mu^{(B)} = 1$.

Equation (6) gives $4 \mid m_B$. We now regard the sum over β. Given β define $\tilde{\beta}$ in the system of representatives we are summing over in terms of $\tilde{\beta} \equiv \beta - \frac{N\nu}{2B} \ \left(\frac{N\nu}{B}\right)$.

We see $\left(\tilde\beta, \frac{N\nu}{B}\right) = 1$ and $\epsilon_\beta = \epsilon_{\tilde\beta}$. Since β and β^* are odd a direct calculation yields

$$\tilde\beta^* \equiv \beta^* + \frac{N\nu}{2B} \quad \left(\frac{N\nu}{B}\right) \qquad \text{for any } \tilde\beta^* \text{ such that } \tilde\beta\tilde\beta^* \equiv 1 \left(\frac{N\nu}{B}\right).$$

Observing that due to $4 \mid m_B$ we have

$$\frac{B\mu^{(B)*2}\frac{N\nu}{2B}}{N\nu} m_B \in \mathbb{Z},$$

the exponential subexpression is the same for β and $\tilde\beta$. By (4.5) on page 24

$$\overline{\chi}\left(\frac{B}{\nu}\beta - \frac{C}{\nu}\gamma\right) = -\overline{\chi}\left(\frac{B}{\nu}\tilde\beta - \frac{C}{\nu}\gamma\right).$$

Since β and $\tilde\beta$ are distinct and $\tilde{\tilde\beta} = \beta$ we then see that the sum over β vanishes. Hence we may always assume $2\mu^{(C)}\nu \nmid C$ as well as $2\mu^{(B)}\nu \nmid B$. In particular we have $\mu^{(B)} = \mu^{(C)} = 1$.

We set

$$\delta = (B, C),\ b = \frac{B}{\delta},\ c = \frac{C}{\delta}.$$

Then by (6) we find $m_B = c^2 m$, $m_C = b^2 m$ for one $m \in \mathbb{N}$. We may assume $\delta \mid \nu$ since otherwise $\overline{\chi}\left(\frac{B\beta - C\gamma}{\nu}\right)$ vanishes.

We consider

$$\sum_{\substack{\beta\left(\frac{N\nu}{\delta b}\right)^\times \\ \gamma\left(\frac{N\nu}{\delta c}\right)^\times \\ \delta b\beta \equiv \delta c\gamma\ (\nu)}} \overline{\chi}\left(\frac{\delta b\beta - \delta c\gamma}{\nu}\right) f_{b^2m,\delta c,d,\lambda,\epsilon,\epsilon_\gamma}(Z_1, w)\overline{g_{c^2m,\delta b,d,\lambda,\epsilon,\epsilon_\beta}(Z_1, w)} e^{2\pi\imath\frac{m\delta bc(b\gamma^* - c\beta^*)}{N\nu}p^*}$$

$$= \sum_{\substack{\beta'\left(\frac{N\nu}{\delta bc}\right)^\times \\ \gamma'\left(\frac{N\nu}{\delta bc}\right)^\times}} \sum_{\substack{\beta\left(\frac{N\nu}{\delta b}\right)^\times \\ \gamma\left(\frac{N\nu t}{\delta c}\right)^\times \\ \beta - \beta' \equiv \gamma - \gamma' \equiv 0\ \left(\frac{N\nu}{\delta bc}\right) \\ \delta b\beta \equiv \delta c\gamma\ (\nu)}} \overline{\chi}\left(\frac{\delta b\beta - \delta c\gamma}{\nu}\right) f_{b^2m,\delta c,d,\lambda,\epsilon,\epsilon_\gamma}(Z_1, w)$$

$$\cdot \overline{g_{c^2m,\delta b,d,\lambda,,\epsilon,\epsilon_\beta}(Z_1, w)} e^{2\pi\imath\frac{m\delta bc(b\gamma^* - c\beta^*)}{N\nu}p^*}.$$

Observing that

$$\delta bc = \prod_{q \mid BC,\ q \text{ prime}} q^{\max\{\nu_p(B),\, \nu_p(C)\}}$$

we see $\nu \mid \frac{N\nu}{\delta bc}$ and thus if $\beta \equiv \beta' \left(\frac{N\nu}{\delta bc}\right)$ we have $\epsilon_\beta = \epsilon_{\beta'}$ and analogously for γ. Moreover the constraint $\delta b\beta \equiv \delta c\gamma\ (\nu)$ may be imposed only on β' and γ'.

We now treat the exponential expression. Observe $\beta \equiv \beta' \left(\frac{N\nu}{\delta bc}\right)$ if and only if $\beta^* \equiv \beta'^* \left(\frac{N\nu}{\delta bc}\right)$ and analogously for γ. We have

$$\frac{m\delta bcb\frac{N\nu}{\delta bc}}{N\nu} = bm \in \mathbb{Z},$$
$$\frac{m\delta bcc\frac{N\nu}{\delta bc}}{N\nu} = cm \in \mathbb{Z}.$$

Hence we can consider the inner sum

$$\sum_{\substack{\beta\left(\frac{N\nu}{\delta b}\right)^\times \\ \gamma\left(\frac{N\nu}{\delta c}\right)^\times \\ \beta-\beta'\equiv\gamma-\gamma'\equiv 0\ \left(\frac{N\nu}{\delta bc}\right)}} \overline{\chi}\left(\frac{\delta b\beta - \delta c\gamma}{\nu}\right).$$

If $\delta bc \nmid \nu$ then $N \nmid \frac{N\nu}{\delta bc}$ and using (4.3) on page 23 we see that the sum vanishes.

Thus we may assume $\delta bc \mid \nu$. Then by $\delta b\beta \equiv \delta c\gamma\ (\nu)$ we see $(c,\nu) = (b,\nu) = 1$ hence $b = c = 1$ and $\delta \mid \nu$. If $\nu \neq 1$ the expression $\chi\left(\frac{\beta-\gamma}{\nu/\delta}\right)$ vanishes whenever $\delta = \nu$ since β and γ are odd. Hence we may assume $\delta \neq \nu$ if $\nu \neq 1$.

We consider the sum given above and write it differently. Observe that $\epsilon_\beta = \epsilon_{\beta'}$ whenever $\beta \equiv \beta'\ (\nu)$ and analogously for γ. In addition $\epsilon_\gamma = \epsilon_\beta$ except if $\nu = 4$ and $\delta = 2$. Then the character vanishes if $\epsilon_\gamma = \epsilon_\beta$ and hence we consider

$$\sum_{\substack{\epsilon\in\{\pm 1\} \\ \beta\left(\frac{N\nu}{\delta}\right)^\times \\ \gamma\left(\frac{N\nu}{\delta}\right)^\times \\ \beta\equiv\gamma\ \left(\frac{\nu}{\delta}\right)}} \overline{\chi}\left(\frac{\beta-\gamma}{\frac{\nu}{\delta}}\right) e^{2\pi\imath m\frac{\gamma^*-\beta^*}{N\frac{\nu}{\delta}}p^*}$$
$$\cdot f_{m,\delta,d,\lambda,\epsilon,\epsilon_{\gamma'}}(Z_1,w) \cdot \overline{g_{m,\delta,d,\lambda,\epsilon,\epsilon_{\beta'}}(Z_1,w)}.$$

We have $C\nu, B\nu = \delta\nu \mid N$ and thus by (8.1) on page 55 $f_{m,\dots}$ and $g_{m,\dots}$ only depend on ϵ, ϵ_γ and ϵ_β by means of $\epsilon\epsilon_\gamma$ and $\epsilon\epsilon_\beta$ respectively. Hence we may consider an inner sum.

By (4.7) on page 25 this inner sum vanishes if $\delta \neq 1$. Otherwise it equals

$$\sum_{\epsilon\in\{\pm 1\}} A_{\overline{\chi},\nu}(mp^*) f_{m,\delta,d,\lambda,1,\epsilon}(Z_1,w) \cdot \overline{g_{m,\delta,d,\lambda,1,\epsilon}(Z_1,w)}.$$

We now see $B\nu = C\nu = \nu \mid N$ and by (8.1) on page 55 we know $f_{m,\delta,d,\lambda,1,\epsilon}$ and $g_{m,\delta,d,\lambda,1,\epsilon}$. These functions depend only on d and λ by means of $d\lambda_1$ and $d\lambda_2$. Hence we may define $\Lambda = \{1,\ldots,\nu\}^2$, use the periodicity of f_m and g_m. Observe that since S is a finite sum of cusp forms the integral over the Fourier series converges absolutely and we may reorder them.

$$
\begin{aligned}
&\left(\pi^{-s}N^{2s}p^{\frac{3}{2}s}(1+p^{-s})\Gamma(s)L(2s,\chi^2)\frac{1}{2i_N\nu^4}G_{\overline{\chi}}^{-1}\right)^{-1}H(s)\\
&= \int\limits_{\substack{Z_1\in\mathcal{F}_1\\ v\in\mathbb{R}/Y_1\mathbb{Z}\\ u\in[-\frac{1}{2},\frac{1}{2}]\\ y>Y_1^{-1}[v]\\ x\in[0,\frac{1}{p}]}} (\det Y_1)^{k-3}(y-Y_1^{-1}[v])^{s+k-3}\sum_{\epsilon\in\{\pm 1\}} G_{\overline{\chi}}S\,|D_\epsilon(Z)dXdY\\
&= \int\limits_{\substack{Z_1\in\mathcal{F}_1\\ v\in\mathbb{R}/Y_1\mathbb{Z}\\ u\in[-\frac{1}{2},\frac{1}{2}]\\ y>Y_1^{-1}[v]\\ x\in[0,\frac{1}{p}]}} (\det Y_1)^{k-3}(y-Y_1^{-1}[v])^{s+k-3}\sum_{\substack{\epsilon\in\{\pm 1\}\\ m\geq 1\\ \lambda\in\Lambda}} A_{\overline{\chi},\nu}(m)\\
&\qquad\cdot f_m\left(Z_1, w+\frac{pp^*\epsilon}{\nu}(Z_1\lambda_1+\lambda_2)\right)e^{2\pi\imath m\left(\frac{pp^*\epsilon}{\nu}\right)^2(Z_1\lambda_1^2+\lambda_1\lambda_2)}e^{2\pi\imath m\frac{2pp^*\epsilon}{\nu}w\lambda_1}e^{2\pi\imath mz}\\
&\qquad\overline{\cdot g_m\left(Z_1, w+\frac{pp^*\epsilon}{\nu}(Z_1\lambda_1+\lambda_2)\right)e^{2\pi\imath m\left(\frac{pp^*\epsilon}{\nu}\right)^2(Z_1\lambda_1^2+\lambda_1\lambda_2)}e^{2\pi\imath m\frac{2pp^*\epsilon}{\nu}w\lambda_1}e^{2\pi\imath mz}}\\
&\qquad dX_1dY_1dudvdxdy\\
&= \frac{1}{p}\int\limits_{\substack{Z_1\in\mathcal{F}_1\\ v\in\mathbb{R}/Y_1\mathbb{Z}\\ u\in[-\frac{1}{2},\frac{1}{2}]\\ y>Y_1^{-1}[v]}} (\det Y_1)^{k-3}(y-Y_1^{-1}[v])^{s+k-3}\sum_{\substack{\epsilon\in\{\pm 1\}\\ m\geq 1\\ \lambda\in\Lambda}} A_{\overline{\chi},\nu}(m)\\
&\qquad\cdot f_m\left(Z_1, w+\frac{pp^*\epsilon}{\nu}(Z_1\lambda_1+\lambda_2)\right)\overline{g_m\left(Z_1, w+\frac{pp^*\epsilon}{\nu}(Z_1\lambda_1+\lambda_2)\right)}\\
&\qquad\cdot e^{-4\pi m\left(\frac{pp^*\epsilon}{\nu}\right)^2Y_1\lambda_1^2}e^{-4\pi m\frac{2pp^*\epsilon}{\nu}v\lambda_1}e^{-4\pi my}dX_1dY_1dudvdy
\end{aligned}
$$

$$= \frac{1}{p}\Gamma(s+k-2)(4\pi)^{-s-k+2} \int\limits_{\substack{Z_1\in\mathcal{F}_1\\ v\in\mathbb{R}/Y_1\mathbb{Z}\\ u\in[-\frac{1}{2},\frac{1}{2}]}} (\det Y_1)^{k-3} \sum_{\substack{\epsilon\in\{\pm1\}\\ m\geq 1\\ \lambda\in\Lambda}} A_{\overline{\chi},\nu}(m)m^{-s-k+2}$$

$$\cdot f_m\left(Z_1, w+\frac{pp^*\epsilon}{\nu}(Z_1\lambda_1+\lambda_2)\right)\overline{g_m\left(Z_1, w+\frac{pp^*\epsilon}{\nu}(Z_1\lambda_1+\lambda_2)\right)}$$

$$\cdot e^{-4\pi m\left(\frac{pp^*\epsilon}{\nu}\right)^2 Y_1\lambda_1^2} e^{-4\pi m\frac{2pp^*\epsilon}{\nu}v\lambda_1} e^{-4\pi m Y_1^{-1}[v]} dX_1 dY_1 du dv$$

$$= \frac{1}{p}\Gamma(s+k-2)(4\pi)^{-s-k+2} \int\limits_{\substack{Z_1\in\mathcal{F}_1\\ v\in\mathbb{R}/Y_1\mathbb{Z}\\ u\in[-\frac{1}{2},\frac{1}{2}]}} (\det Y_1)^{k-3} \sum_{\substack{\epsilon\in\{\pm1\}\\ m\geq 1\\ \lambda\in\Lambda}} A_{\overline{\chi},\nu}(m)m^{-s-k+2}$$

$$\cdot f_m(Z_1,w)\overline{g_m(Z_1,w)}$$

$$\cdot e^{-4\pi m\left(\frac{pp^*\epsilon}{\nu}\right)^2 Y_1\lambda_1^2} e^{-4\pi m\frac{2pp^*\epsilon}{\nu}\left(v-\frac{pp^*\epsilon}{\nu}Y_1\lambda_1\right)\lambda_1}$$

$$e^{-4\pi m Y_1^{-1}[v-\frac{pp^*\epsilon}{\nu}Y_1\lambda_1]} dX_1 dY_1 du dv$$

$$= \frac{1}{p}\Gamma(s+k-2)(4\pi)^{-s-k+2} \int\limits_{\substack{Z_1\in\mathcal{F}_1\\ v\in\mathbb{R}/Y_1\mathbb{Z}\\ u\in[-\frac{1}{2},\frac{1}{2}]}} (\det Y_1)^{k-3} \sum_{\substack{\epsilon\in\{\pm1\}\\ m\geq 1\\ \lambda\in\Lambda}} A_{\overline{\chi},\nu}(m)m^{-s-k+2}$$

$$\cdot f_m(Z_1,w)\overline{g_m(Z_1,w)} e^{-4\pi m Y_1^{-1}[v]} dX_1 dY_1 du dv$$

$$= \frac{1}{p}\Gamma(s+k-2)(4\pi)^{-s-k+2} 4\nu^2 \sum_{m\geq 1} A_{\overline{\chi},\nu}(m)m^{-s-k+2}\langle f_m,\, g_m\rangle$$

$$= \frac{1}{p}\Gamma(s+k-2)(4\pi)^{-s-k+2} 4\nu^2 A_{\overline{\chi},\nu}(1) D_{f,g,\overline{\chi}}(s+k-2).$$

Using (4.7) on page 25 we get

$$H(s) = \pi^{-s}N^{2s}p^{\frac{3}{2}s}(1+p^{-s})\Gamma(s)L(2s,\chi^2)\frac{1}{2i_N\nu^4}G_{\overline{\chi}}^{-1}$$

$$\frac{1}{p}\Gamma(s+k-2)(4\pi)^{-s-k+2}4\nu^2 A_{\overline{\chi},\nu}(1) D_{f,g,\overline{\chi}}(s+k-2)$$

$$= \mu(R)\overline{\chi}^{(2)}(R) G_\chi^4 G_{\overline{\chi}^{(2)}} p^{\frac{3}{2}s-1}(1+p^{-s})$$

$$\frac{2}{i_N N^2}(4\pi)^{-s-k+2}\pi^{-s}N^{2s}L(2s,\chi^2)\Gamma(s)\Gamma(s+k-2)D_{f,g,\overline{\chi}}(s+k-2)$$

$$= \mu(R)\overline{\chi}^{(2)}(R) G_\chi^4 G_{\overline{\chi}^{(2)}} p^{\frac{3}{2}s-1}(1+p^{-s})$$

$$\frac{2}{i_N N^2}\left(\frac{\pi}{N^2}\right)^{k-2}\mathbb{D}_{f,g,\overline{\chi}}(s+k-2) \qquad \blacksquare$$

§ 9 Frequently used symbols

- $A_{\chi,\nu}(m)$: character sum, p. 25
- C_{θ_0} : a certain sum used for estimations, p. 38
- D_η : matrix, p. 12
- $D_{f,g}$: Rankin convolution of f and g, p. 8
- $\mathbb{D}_{f,g}$: completed Rankin convolution of f and g, p. 8
- $D_{f,g,\chi}$: twisted Rankin convolution of f and g, p. 8
- $\mathbb{D}_{f,g,\chi}$: twisted and completed Rankin convolution of f and g, p. 8
- $d\nu_n$: invariant measure on $\mathbb{H}_n$, p. 4
- $d\nu_n^*$: invariant measure on $\mathbb{H}_{n-1} \times \mathbb{C}^{n-1}$, p. 4
- $E^{(p)*}_{Np,N^2/\kappa p,\chi}(Z,s)$: Eisenstein series, p. 34
- $\mathbb{E}^{(p)*}_{Np,N^2/\kappa p,\chi}(Z,s)$: Eisenstein series, p. 34
- $\mathcal{E}^{(p)*}_{Np,N^2/\kappa p,q,r,\chi}(\cdot,s)$: difference of Eisenstein series, p. 47
- $\mathcal{F}_{n,1}$: fundamental domain for $\Gamma_{n,1}$, p. 12
- $\mathcal{F}_n^*$: fundamental domain for the Jacobi group, p. 4
- $\mathcal{F}^{(t)}_{n,1}$: fundamental domain for $\Gamma^{(t)}_{n,1}$, p. 13
- f_χ : twist of a modular form, p. 29
- Γ_n : Siegel modular group, p. 4
- $\Gamma_{n,1}(N, N^2/\kappa)$: subgroup of Γ_n, p. 12
- $\Gamma_{n,1}$: subgroup of Γ_n, p. 12
- $\Gamma^1_{n,1}(N, N^2/\kappa)$: subgroup of Γ_n, p. 12
- $\Gamma^{(t)}_{n,1}(N\tau, (N\tau)^2/\kappa)$: paramodular group, p. 13
- $\Gamma^{(t)}_{n,1}$: parabolic subgroup of the paramodular group, p. 13
- $\Gamma^{(p)*}_{2,1}(Np, N^2/\kappa p)$: extension of the paramodular group, p. 15
- G_χ : Gauß sum, p. 4
- $\mathbb{H}_n$: Siegel upper half space, p. 4
- $H_p(N)$: matrix, p. 15
- $L(s,\chi)$: L-function, p. 4
- $\mu(\cdot)$: Möbius function, p. 4
- M_η : matrix, p. 12
- $M_{d,\gamma}$: matrix, p. 19
- M_λ : matrix, p. 12
- M^{tr} : transposed matrix, p. 4
- M_1 : upper left submatrix, p. 4

- $\nu_p(\cdot)$: p-adic exponential valuation, p. 4
- $\phi(\cdot)$: Euler ϕ-function, p. 4
- $P_{d,t}$: matrix, p. 12
- P_Z : matrix induced by an element of the Siegel upper half space, p. 4
- $\mathrm{Sp}_n(\mathbb{R})$: symplectic matrices with entries in $\mathbb{R}$, p. 4
- $\mathrm{tr}(M)$: trace of a matrix, p. 4
- W_η : matrix, p. 12
- $\zeta(s,u,v,P)$: Epstein ζ-function, p. 4
- $\zeta^*(s,u,v,P)$: completed Epstein ζ-function, p. 4
- $\mathbb{Z}^{n\times}$: primitive columns of length n, p. 4
- $\mathbb{1}_N$: trivial character with period N, p. 4
- $.^{(m,n)}$: matrices of height m and width n, p. 4
- $|_k\cdot$: slash operator, p. 4
- $[\Gamma', k, \chi]$: set of Siegel modular forms, p. 4
- $[\Gamma', k, \chi]_0$: set of Siegel cusp modular forms, p. 4
- $\{\cdot,\cdot\}$: scalar product for Siegel modular forms, p. 4
- $\langle\cdot,\cdot\rangle$: scalar product for Jacobi modular forms, p. 4

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